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BOSTON UNIVERSITY

GRADUATE SCHOOL

Dissertation

APPLICATION OF THE LORENTZ TRANSFORMATION TO THE  
GENERAL SECOND ORDER SOLUTION OF THE  
PROBLEM OF ETHER DRIFT MEASUREMENT

by

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(B.S. in Ed., Boston University, 1934)

(A.M., Boston University, 1935)

submitted in partial fulfilment of the  
requirements for the degree of  
Doctor of Philosophy

1939

# FALCON BOND

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## INTRODUCTION

The theory that light consists of a wave motion in a luminiferous ether was once generally accepted by physicists. According to this theory the ether is supposed to possess the property whereby waves of light may be transmitted and all other optical phenomena may be accounted for. It is assumed that all space, even that occupied by material bodies, is pervaded by this ether, and yet, it permits all bodies to move through it with free motion.

The "ether drift" experiment was originally conceived as an experimental means of ascertaining the actual validity of this hypothesis. The originator of this idea was James Clerk Maxwell in his article on "Ether" in the 1878 edition of the Encyclopedia Britanica. In this article he pointed out that, on the assumption that there is a stationary ether through which the earth is moving freely with an absolute velocity and that light waves are propagated in this free ether in any direction and always with the same velocity relative to the ether; the apparent velocity of light would depend upon whether the light is travelling along the direction of the motion of the earth or at right angles to it. The former path would be slightly lengthened as compared with the latter. Thus a relative motion between the earth and the stationary ether, that is, an "ether drift" should be detectable.

Since the velocity of the earth in its orbit is thirty kilometers per second and the velocity of light is ten thousand times as great, the velocity measured in the line of motion would differ from the velocity at

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right angles to this line by thirty kilometers per second, that is, by one part in ten thousand. This would be a "first order effect". However, since all experimental methods of determining this effect require that the light shall travel from one point to another and back again, a positive effect of the earth's motion on the ray in the outgoing path would be nullified by the opposite effect on the returning ray, except for a "second order effect" due to the motion of the observer during the time of passage of the light. Concerning this effect, Maxwell concluded that "The change in the time of transmission of the light on account of a relative velocity of the ether equal to that of the earth in its orbit would be only one hundred-millionth part of the whole time of transmission and would therefore be quite insensible".

In 1880, while attending the University of Berlin, the late Professor Albert A. Michelson invented the interferometer which bears his name and is generally considered as being especially suited for detecting the hypothesized ether drift-- if it exist. By means of this apparatus, light incident upon a half-silvered plane parallel glass plate, is divided into two beams so that each travels along a separate path in a different direction. After having traversed their respective paths, each ray is reflected back again to the half-silvered mirror where they are again united and sent out in directions determined by the inclinations of the two mirrors. These inclinations are such that the final rays are almost parallel. If the two paths are optically equal in length, the reunited beams of light will blend and there will be no interference. If however, one path is longer than the other, there will result differences of phase which will give rise to interference fringes. As the interferometer is rotated, with reference to the earth, first one arm has a relatively longer path and than the other; so that the interference fringes



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which are actually due to the phase relationships between the two wave fronts of the final reunited rays will undergo slight shifts. The relation is such that an effective change of one wave length in the length of one path as compared with the other will result in a movement of the fringes equal to one fringe width.

The first experiment with this apparatus was carried out by A. A. Michelson in 1881. Later, Michelson together with Professor Edward W. Morley of Western Reserve University, as well as several other scientists, have at various times constructed interferometers and conducted experiments for the detection of this expected shift of fringes. In the July 1933, issue of the Reviews of Modern Physics, Professor Dayton C. Miller presents a summary of the results of these investigators together with a very thorough analysis of extensive investigations of his own. The conclusion at which he arrived is that there exists a small, actual, periodic fringe shift which however is much too small to be accounted for by the hypothesis of a stationary ether.

The Michelson-Morley experiment with its null result suggested that either the existing theory of the ether was incorrect or that it was at least incomplete.

In order to account for the absence of any effect of the earth's translation, Prof. G. F. FitzGerald in 1891 put forth a theory which was put on a firm mathematical basis by H. A. Lorentz in 1895. According to this theory, the dimensions of a solid body undergo slight changes of the order of  $(v/c)^2$  when it moves through the ether. The negative result of the experiment is thus accounted for by this contraction.

Following the contraction theory, the next theory introduced to

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account for the null effect of the "ether drift" experiment, came in 1905 with Professor Albert Einstein's Special Theory of Relativity according to which a material body suffers a contraction in the direction of its motion. Upon the basis of this theory the "ether drift" experiment should have a null result. However, because of the definite positive effect obtained by the very extensive and thorough work of D. C. Miller, even though this effect is much smaller than that originally predicted; an hypothesis which leads to a null effect is no more gratifying than the original stationary ether hypothesis. Nevertheless, in examining the broader postulates<sup>1</sup> of Professor Einstein's General Theory of Relativity announced in 1915, it will be observed that they may be modified so as not to exclude the possibility of a relative velocity of drift. Indeed such triumphs of the General Theory as: the prediction of the bending of a ray of light when passing through a strong gravitational field, accounting for the advance of the perihelion of mercury and predicting that when the source of light is in the atmosphere of a massive star the wave length becomes slightly greater; are of such significance that they have been widely accepted as confirming the General Theory. On the other hand, the Special Theory with its prediction of a null effect may not actually be in violation of the experimental results of the ether drift experiment since it is a special case of the General Theory. The investigation of Michelson and Morley and also the later and more extensive work of D. C. Miller were carried out by means of equal arm interferometers in which the two beams of light traveled along paths which were at right angles to each other.

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1. Eddington, A. S., "The Mathematical Theory of Relativity" p. 104, (1930)

account for the null effect of the "ether drift" experiment, came in 1905 with Professor Albert Einstein's Special Theory of Relativity according to which a material body follows a geodesic in the direction of its motion. Upon the basis of this theory the "ether drift" experiment should have a null result. However, because of the delicate positive effect obtained by the very sensitive and thorough work of H. A. Miller, even though this effect is much smaller than that originally predicted, an hypothesis which leads to a null effect is no more satisfying than the original stationary ether hypothesis. Nevertheless, in examining the broader postulates of Professor Einstein's General Theory of Relativity announced in 1915, it will be observed that they may be modified so as not to exclude the possibility of a relative velocity of drift. Indeed such a change of the General Theory as the prediction of the bending of a ray of light when passing through a strong gravitational field, accounting for the advance of the perihelion of mercury and predicting that when the wave of light is in the atmosphere of a massive star the wave length becomes slightly greater; are of such significance that they have been widely accepted as confirming the General Theory. On the other hand, the Special Theory with its prediction of a null effect may not actually be in violation of the experimental results of the ether drift experiment since it is a special case of the General Theory. The investigation of Michelson and Morley and also the later and more extensive work of D. C. Miller were carried out by means of equal arm interferometers in which the two beams of light travelled along paths which were at right angles to each other.



At a meeting of the American Physical Society held at Wellesley College in the fall of 1937, W. B. Cartmel presented a paper in which he claimed that for a maximum fringe shift the arms should be set at an angle of forty five degrees. With this in view, E. E. Haskins carried out an investigation in which the arms were assumed to be set at this forty five degree angle. He arrived at the conclusion that the fringe shift was even greater than that predicted by Cartmel's formula. In fact he found, that in least one respect Cartmel's formula was in error.

The foregoing remarks are intended to make clear the necessity of carrying out a careful investigation of the entire problem with the view of determining the exact effect to be expected.

In this paper an investigation is carried out in which the lengths of the paths traversed by the two beams are made unequal. Also, the angle between the two arms is assumed to be arbitrary so that the final solution will be entirely general and therefore applicable to the conditions given in any particular experiment.

All of the equations involved in the problem will be set up from the point of view of an observer fixed in the ether. To make the results applicable, the Lorentz transformations will be applied so that the final result will come out in terms of the observer moving with the earth.

Thus, it became possible to turn the interferometer to different angles while observations were in progress. To obtain the required sensitivity, the effective light path was increased by reflecting the light back and forth so that it traversed the diagonal of the square glass block.

At a meeting of the American Physical Society held at "Ellisley College" in the fall of 1937, W. B. Gurnea presented a paper in which he claimed that for a maximum fringe shift the arms should be set at an angle of forty five degrees. With this in view, E. B. Maskin carried out an investigation in which the arms were assumed to be set at this angle. He arrived at the conclusion that the fringe shift was even greater than that predicted by Gurnea's formula. In fact he found that in fact one respect Gurnea's formula was in error.

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In this paper an investigation is carried out in which the lengths of the paths traversed by the two beams are made unequal. Also, the angle between the two arms is assumed to be arbitrary so that the final relation will be entirely general and therefore applicable to the conditions given in any particular experiment.

All of the questions involved in the problem will be set up from the point of view of an observer fixed in the ether. To make the results applicable, the Lorentz transformations will be applied so that the final result will come out in terms of the observer moving with the earth.



### REVIEW OF THE WORK OF OTHERS

The history of the "Ether-Drift" experiment had its inception when A. A. Michelson, while yet a student in Germany conducted experiments both in Berlin and in Potsdam in the years 1880-1881. These were the first attempts made to detect the second-order effect of the relative motion of the earth and the ether. The original attempts of Michelson<sup>1</sup> were unsuccessful. He observed approximately one-hundredth of the effect for which he was looking. This null effect however, was attributable, at least partially, to the fact that the light path was only a little more than one meter in length.

In 1881 Michelson was appointed to the professorship of physics in the Case School of Applied Science in Cleveland where he became acquainted with Professor Edward W. Morley, professor of chemistry in the neighboring Western Reserve University. With new developments in the interferometer and in the method of using it, Michelson, together with Morley, again attempted to detect the then expected effect. In order to avoid disturbance of vibration and distortion, the optical parts of the instrument were mounted on a solid block of sandstone which was floated in mercury contained in a circular tank of cast iron. Thus, it became possible to turn the interferometer to different azimuths while observations were in progress. To obtain the required sensitivity, the effective light path was increased by reflecting the light back and forth so that it traversed the diagonal of the square stone block

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1. Michelson A. A., Am. J. Sci. (3) 28, 120 (1880)

## REVIEW OF THE WORK OF OTHERS

The history of the "Stokes-Drift" experiment and the inception when A. A. Michelson, while yet a student in Germany conducted experiments both in Berlin and in Potsdam in the years 1880-1881. These were the first attempts made to detect the second-order effect of the relative motion of the earth and the ether. The original attempts of Michelson were unsuccessful. He observed approximately one-fourth of the effect for which he was looking. This null effect however, was attributed, at least partially, to the fact that the light path was only a little more than one meter in length.

In 1881 Michelson was appointed to the professorship of physics in the Ohio School of Applied Science in Cleveland where he became acquainted with Professor Edward W. Morley, professor of chemistry in the neighboring Western Reserve University. With new developments in the interferometer and in the method of using it, Michelson, together with Morley, again attempted to detect the then expected effect. In order to avoid disturbance of vibration and distortion, the optical parts of the instrument were mounted on a solid block of granite which was fixed in masonry embedded in a circular hole of cast iron. Thus, it became possible to turn the interferometer to different positions while observations were in progress. To obtain the required sensitivity, the effective light path was increased by reflecting the light back and forth so that it traversed the diagonal of the square stone block



eight times, giving the effect of an interferometer with an arm of approximately 1100 centimeters in length. The series of observations made by Michelson and Morley were of six hours duration and were taken during six different intervals of one hour each. The series of observations consisted of thirty six turns of the interferometer, sixteen readings being made at equidistant points of each turn. This relatively short series of observations showed that the effect was neither of the expected magnitude nor of zero value. The conclusions of this experiment<sup>1</sup>, published in 1887 stated that the observed relative motion of the earth and the ether was not greater than one-fourth of the earth's orbital velocity.

In an address delivered in 1900 at the International Congress of Physics held in conjunction with the International Exposition in Paris, Lord Kelvin explained the significance of the Michelson-Morley experiments as related to the theories of the ether and he expressed the belief that further, more refined experimental work in this field was necessary. Accordingly E. W. Morley and D. C. Miller, who were present at this conference, constructed an interferometer having a light path more than three times as long as that used by Michelson and Morley in 1887. Originally, they employed a framework of heavy lumber but this was soon abandoned in favor of one made of structural steel and floated in a tank of mercury.

In the years 1900 to 1907<sup>2</sup> numerous observations were made; first, in the basement of the Case School of Applied Science, and later, on Euclid

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1. Michelson A. A. and Morley E. W., Am. J. Sci. (3) 34, 333 (1887).

2. Morley E. W. and Miller D. C., Phil. Mag. (6) 9, 680 (1905).

eight times, giving the effect of an interferometer with an arm of approximately 1100 centimeters in length. The series of observations made by Nicholson and Morley were of six hours duration and were taken during six different intervals of the hour. The series of observations consisted of twenty six hours of the interferometer, sixteen readings being made at regular intervals of each hour. The relatively short series of observations showed that the effect was neither of the expected magnitude nor of the value. The conclusions of this experiment, published in 1887 stated that the observed relative motion of the earth and the ether was not greater than one-fourth of the earth's orbital velocity.

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1. Nicholson A. A. and Morley E. W. Am. J. Sci. (3) 24, 232 (1887).

2. Morley E. W. and Miller F. G. Phil. Mag. (6) 1, 600 (1905).



Heights, Cleveland. This latter site is at an altitude of 285 meters. The results obtained showed a very definite positive effect slightly larger than that previously obtained but still too small to be reconciled with the expected effect. At this time Professor Morley retired from active work and the work was continued by D. C. Miller in 1921.

Through the courtesy of the Carnegie Institute the ether-drift interferometer was set up on the grounds of the Mount Wilson Observatory in March 1921. Later the apparatus was returned to Cleveland where numerous observations and improvements were made during the years 1922 and 1923. In 1924, it was again returned to Mount Wilson where there were employed many refinements such as : improved mirror mountings, protection from heat, improved light source, large viewing telescope and other refinements which were developed in the laboratory tests at Cleveland.

Over two hundred thousand readings were taken with this instrument. This represents the most exhaustive study in this field. The analysis of the results of this work, together with a rather lengthy review of the whole topic in general and his own work in particular, are given by D. C. Miller in the Review of Modern Physics<sup>1</sup>. Miller reaches the conclusion that there does exist a small positive fringe shift equal to about five per cent of the expected effect.

Among other investigators who have studied this problem, we find R. J. Kennedy and E. M. Thorndike<sup>2</sup> who carried out experiments with an interferometer of a different design. The arms of their apparatus were made

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1. Miller D. C. Rev. Mod. Phys., 5, 203, (1933).

2. Kennedy R. J. and Thorndike E. M. Phys. Rev., 42, 400, (1932).

heights, Cleveland. This latter site is at an altitude of 288 meters. The results obtained showed a very definite positive effect slightly larger than that previously obtained but still too small to be reconciled with the expected effect. At this time Professor Miller returned from active work and the work was continued by D. C. Miller in 1931.

Through the courtesy of the Carnegie Institute the other half of the interferometer was set up on the grounds of the Mount Wilson Observatory in March 1931. Later the apparatus was returned to Cleveland where numerous observations and improvements were made during the years 1932 and 1933. In 1934 it was again returned to Mount Wilson where there were employed many refinements such as: improved mirror mountings, protection from heat, improved light source, large viewing telescope and other refinements which were developed in the laboratory tests at Cleveland.

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1. Miller D. C. Rev. Mod. Phys., 3, 203 (1933).
  2. Kennedy E. V. and Thorndike E. M. Phys. Rev., 42, 400 (1933).



unequal in length and were set at such an angle to each other that the light reflected from the plate would be plane polarized. This angle they found to be sixty-five degrees. However, it does not seem that the investigators expected that the effect would be any greater by choosing that angle. The length of the light path in their apparatus was relatively short and they reported an extremely small effect, if any. In their report, they develop a theory based on the Lorentz-FitzGerald contraction, from which they conclude that no effect should be expected, regardless of the magnitude of the angle between the arms.

In 1926<sup>1</sup> and 1927<sup>2</sup>, A. Piccard and E. Stahel reported results of experiments automatically performed and recorded in balloons at an altitude of 1900 meters. According to these experiments, there was a positive effect in rather close agreement with the results of D. C. Miller.

Professor Georg Joos<sup>3</sup>, in Jena, working with an interferometer, mounted on a quartz base suspended in an evacuated metal housing and provided with photographic registration obtained results indicating that any existing ether drift could not exceed one kilometer per second.

The theoretical phase of the ether drift problem has been treated by a few investigators. In a paper by W. M. Hicks<sup>4</sup> of University College, Sheffield, there is an important theoretical investigation of the original experiment. In his work, Hicks considered, in particular, the effect in an

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<sup>1</sup> Piccard, A. and Stahel, E., Comptes Rendus 183, 420, (1926).

<sup>2</sup> Piccard, A. and Stahel, E., Comptes Rendus 185, 1198, (1927).

<sup>3</sup> Joos, Georg, Ann. D. Physik (5), 7, 385, (1930).

<sup>4</sup> Hicks, W. M., Phil. Mag., 3, 9, (1902).





interferometer of which the arms are at right angles to each other and the normals to the mirrors make a slight angle with the arms. From his investigation he concluded that this slight inclination introduces a small first order effect which is singly periodic in a complete rotation of the interferometer. The amplitude of this full period effect varies inversely as the width of the fringes being used at the time of the observations. Miller states that this effect was present in all the observations including the original observations of Michelson and Morley.

H. A. Lorentz in his book "The Theory of the Electron"<sup>1</sup> develops a theory that a body contracts in the direction of its motion. He discusses the ether drift experiment and derives the well known result  $2\frac{v^2}{c^2}$ .

The theory of the ether drift experiment was also discussed by A. Righi<sup>2,3</sup>. He concluded that there should be no fringe shift detectable, because, on account of the rotation, the two beams exchange places and hence the effect cancels out.

E. R. Hedrick<sup>4</sup> and Paul S. Epstein<sup>5</sup> presented papers dealing mainly with the effect produced when the mirrors are slightly oblique to the arms. In his discussion of the ninety degree interferometer, Epstein states that the angle between the rays in their final path is  $\frac{v^2}{c^2} \cos 2\phi$ . The latter expression corresponds to equation 58 in this paper.

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1. Lorentz H. A., "The Theory of the Electron", B. G. Teubner, Leipzig, (1916)

2. Righi A. , Comptes Rendus, 168, 837, (1919)

3. Righi A. , Comptes Rendus, 170, 499 and 1550, (1920)

4. Hedrick E. R., Astrophys. J., 68, 374, (1928)

5. Epstein P. S., Astrophys. J., 68, 383, (1928)

interferometer of which the arms are at right angles to each other and the normals to the mirrors make a slight angle with the arms. From this it follows that he concluded that this slight inclination introduces a small first order effect which is simply periodic in a complete rotation of the interferometer. The magnitude of this first order effect varies inversely as the width of the fringes being used at the time of the observations. Michelson states that this effect was present in all the observations including the original observations of Michelson and Morley.

H. A. Lorentz in his book "The Theory of the Electron" develops a theory that a body contracts in the direction of its motion. He discusses the ether drift experiment and derives the well known result  $\frac{v}{c}$ . The theory of the ether drift experiment was also discussed by A. N. S. He concluded that there would be no fringe shift detectable because, on account of the rotation, the two beams exchange places and hence the effect cancels out. E. R. Hedrick and Paul A. Epstein presented papers dealing mainly with the effect produced when the mirrors are slightly oblique to the arms. In his discussion of the ninety degree interferometer, Epstein states that the angle between the rays in their final path is  $\frac{v}{c} \cos \theta$ . The latter expression corresponds to equation 36 in this paper.

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1. Lorentz H. A., "The Theory of the Electron", D. B. Teubner, Leipzig, (1911)  
2. Michelson A., "Comptes Rendus", 102, 845, (1911)  
3. Michelson A., "Comptes Rendus", 119, 422 and 1250, (1915)  
4. Hedrick E. R., "Astrophys. J.", 22, 370, (1926)  
5. Epstein P. A., "Astrophys. J.", 22, 388, (1926)



The "Ether Drift Conference"<sup>1</sup> gives evidence of the amount of attention that scientists have given this problem. This conference was held on February 4 and 5, 1927, at the Mount Wilson Observatory in Pasadena, California. The attention of the entire conference was centered upon the problem of measuring the ether drift. Among those present at this conference were Michelson, Lorentz, Miller, Kennedy, Hedrick and Epstein.

W. B. Cartmel<sup>2</sup> wrote a paper claiming that for a maximum effect the arms of an interferometer should be inclined forty-five degrees to each other. The formula he derived gave the expected fringe shift as a function of the angle of drift. However, his formula seems to predict an effect even when the arms are taken parallel to each other. Accordingly, E. E. Haskins<sup>3</sup> made an investigation of the effect that may be expected when the arms are separated by an angle of forty-five degrees. He concluded that the expected maximum displacement of the fringes should be more than twice as great as the maximum displacement obtainable with an interferometer of similar size and arms at right angles to each other.

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1. Ether Drift Conference, *Astrophys. J.*, 68, 352, (1928)
  2. Cartmel, W. B., *Physical Review*, 53, 108, (1938)
  3. Haskins, E. E., *Doctorate Dissertation*, Boston University, (1938)

The "White Wolf Conference" gives evidence of the extent of  
 attention that scientists have given this problem. This conference was held  
 on February 6 and 7, 1957, at the Mount Wilson Observatory in Pasadena,  
 California. The sessions of the entire conference was devoted to the  
 problem of weathering the outer shell, among them were the following  
 were mentioned, namely, Miller, Kennedy, Joditsch and others.  
 W. S. Gardner, who is a paper discussing the use of a neutron effect the  
 kind of an instrument should be a kind of forty-five degrees in each other.  
 The results he obtained gave the expected results which are a function of the  
 angle of reflection. However, his formula seems to predict an effect even when  
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 maximum displacement of the fringes should be more than twice as great as the  
 maximum displacement obtained with an instrument of similar size and  
 type at right angles to each other.

- 1. White Wolf Conference, Pasadena, Feb. 6, 7, 1957 (1957)
- 2. Gardner, W. S., Physics Review, 105, 103, (1953)
- 3. Gardner, W. S., Astrophysical Journal, 117, 101, (1952)



### PRELIMINARY REMARKS CONCERNING FIGURE I

Figure I is drawn entirely from the point of view of an observer fixed in the ether. Relative to this observer, the Michelson Interferometer is moving in a direction  $\underline{v}$ , where the vector  $\underline{v}$  is the horizontal component of the absolute velocity of the earth.

As the apparatus is set up by the moving observer the plate M is made to bisect the angle between the two arms S' and S'' but due to the Lorentz contraction <sup>1</sup> the fixed observer will find that the two angles are not equal hence they have been designated by the different symbols  $\theta$  and  $\theta'$  and set equal after applying the Lorentz transformation.

For the general solution of this problem S' and S'' have been made unequal. But, just as in the case of  $\theta$ 's the two arms may be set equal after the Lorentz transformations have been applied.

The mirror M' is adjusted by the moving observer to be perpendicular to the arm to which it is fixed but due to the Lorentz contraction of the mirror, the fixed observer will find that the angle between the normal to the mirror and the arm S' will be  $\sigma$ .

In order that the fringes observed in the telescope may be of a convenient width, the angle between the mirror M'' and the arm to which it is fixed is made just slightly different from  $\pi/2$ . However,  $\omega$  the angle which the fixed observer measures between the normal to the mirror and S'' is

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1. The Lorentz transformation equations of the various angles and distances discussed in this section, will be derived on pages 18 to 30.

RELATIVISTIC TRANSFORMATIONS

Figure 2 is drawn entirely from the point of view of an observer fixed in the ether. Relative to this observer, the Michelson interferometer is moving in a direction  $x$ , where the vector  $x$  is the horizontal component of the absolute velocity of the earth.

As the apparatus is set up by the moving observer the glass M is

made to bisect the angle between the two arms  $S'$  and  $S''$  but due to the

relativistic contraction the fixed observer will find that the two angles are

not equal since they have been designated by the different speeds  $S'$  and  $S''$

and not equal after applying the Lorentz transformation.

For the general solution of this problem  $S'$  and  $S''$  have been made

unequal. But, just as in the case of  $S'$  the two arms may be set equal

after the relativistic transformation have been applied.

The mirror  $M'$  is adjusted by the moving observer to be perpendicular

to the arm to which it is fixed but due to the relativistic contraction of

the mirror, the fixed observer will find that the angle between the mirror

to the mirror and the arm  $S'$  will be  $\alpha$ .

In order that the things observed in the telescope may be of

a convenient width, the angle between the mirror  $M'$  and the arm to which it

is fixed is made just slightly different from  $90^\circ$ . However, the angle which

the fixed observer measures between the mirror and  $S'$  is

1. The relativistic transformation equations of the various angles and distances

discussed in this section, will be derived on pages 15 to 20.



slightly modified because of Lorentz contraction of mirror  $M''$ .

In the experiment, it is found that the telescope which is equally inclined to the two final rays must be focused on a point  $T$  which is in the vicinity of the mirror  $M'$ . Thus, the two paths of the light that must be compared are  $AC+CH-HT$  and  $AP+PL-LT$ .

Before proceeding with the main problem, it will be desirable to develop all the necessary equations of the Lorentz transformations, also the law of reflection from a moving mirror and the equation of the change of wave length upon reflection.

U.S. GOVERNMENT

OFFICE OF THE SECRETARY OF THE ARMY

WASHINGTON, D.C.

MEMORANDUM FOR THE SECRETARY OF THE ARMY  
SUBJECT: [Illegible]









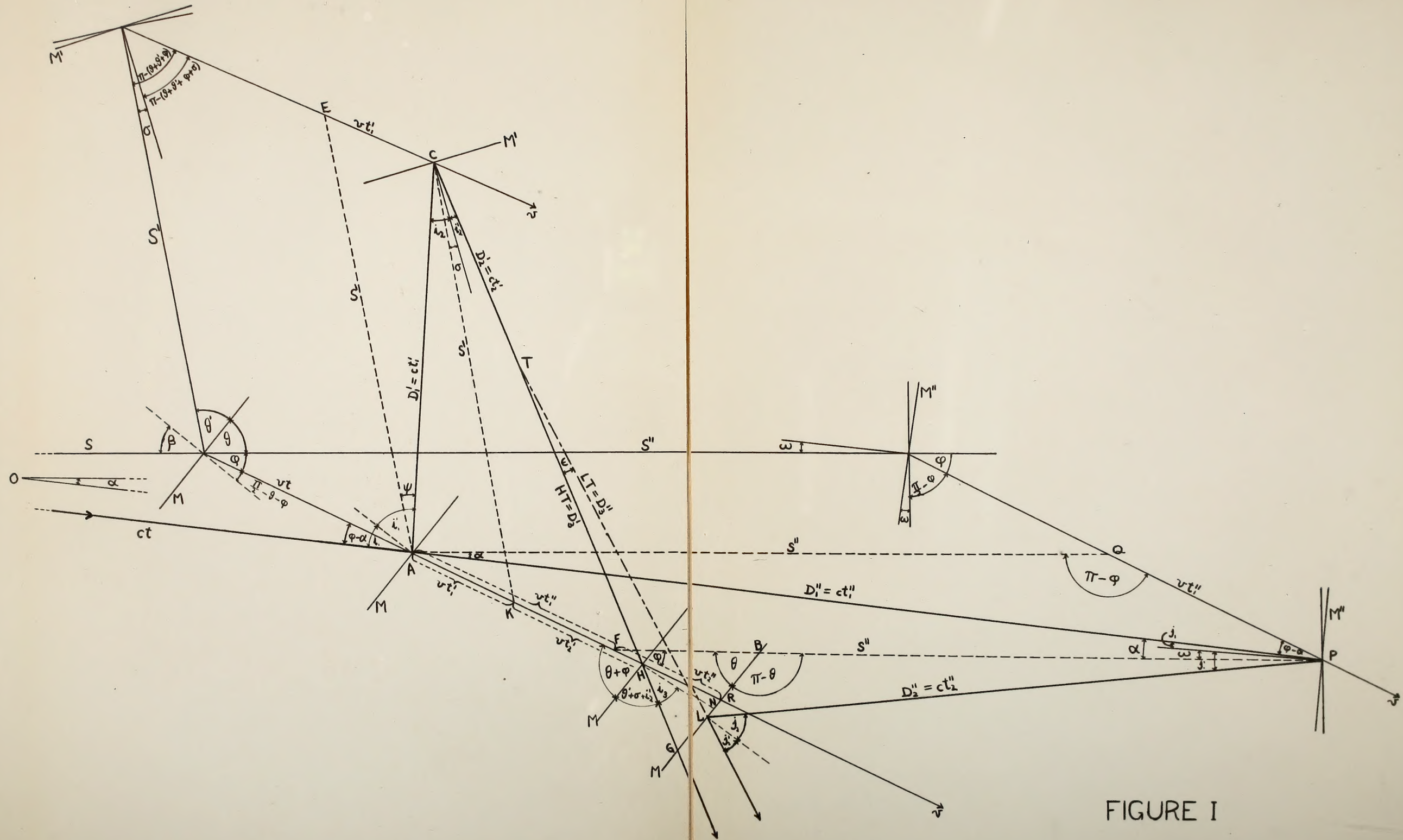
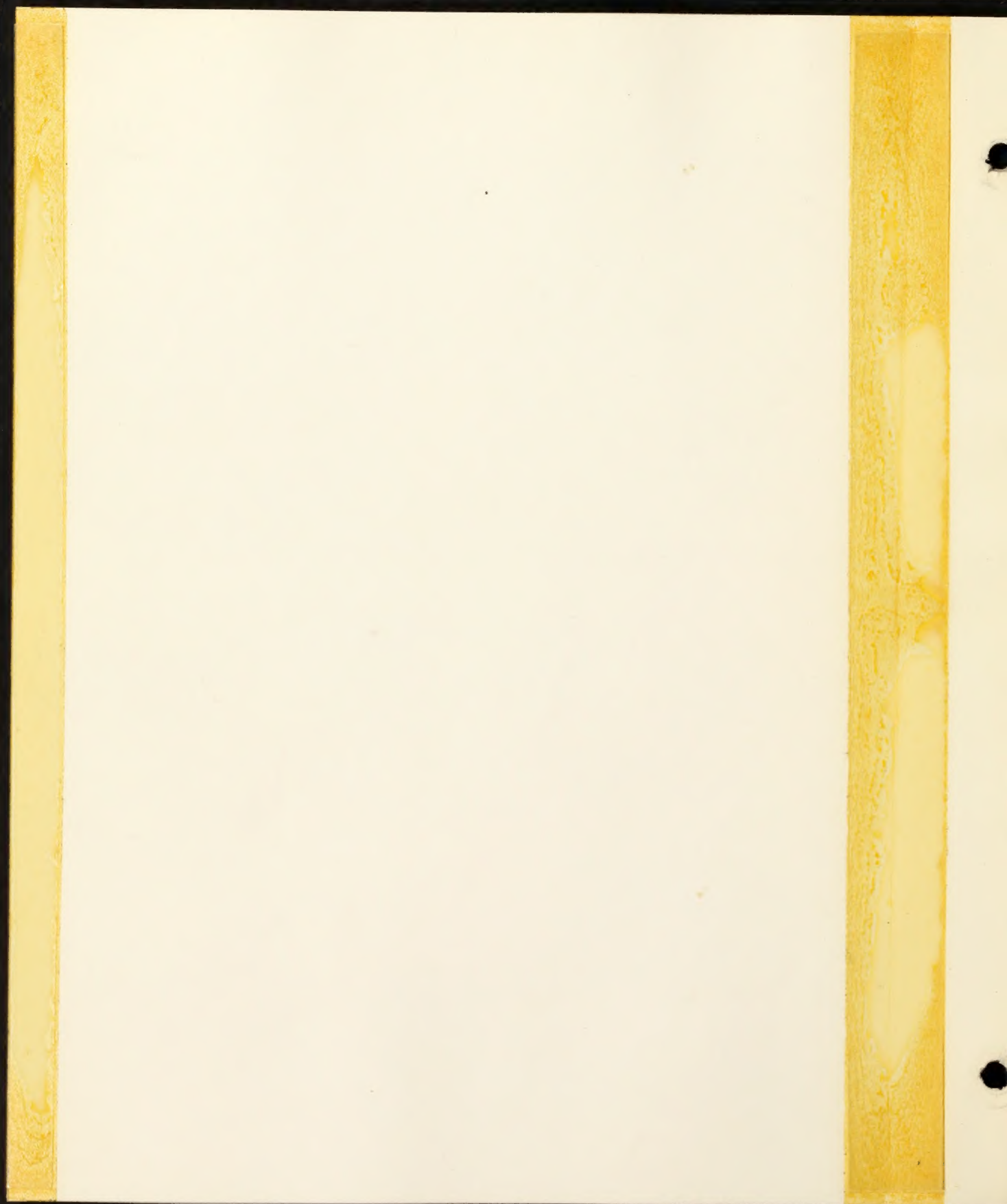


FIGURE I





### LORENTZ TRANSFORMATION

Consider two inertial systems  $S$  and  $S_0$ ,  $S$  being fixed relative to the ether and  $S_0$  being fixed relative to the earth but moving relative to  $S$ . Throughout this work all angles and distances measured in the latter moving system will be designated by the subscript zero, whereas all angles and distances not bearing the subscript zero will refer to the former fixed system.

Let  $XYZ$  represent a rectangular system of coordinates fixed in the system  $S$  and  $X_0Y_0Z_0$  a parallel set fixed in the system  $S_0$ . Let the two systems be oriented so that the velocity  $\underline{v}$  of  $S_0$  relative to  $S$  is in the positive  $X$  direction. Denote the coordinates and time as measured by an observer in  $S$  as  $x, y, z, t$  and as measured by observer in  $S_0$  as  $x_0, y_0, z_0, t_0$ . Let the zero of time be determined by taking  $t=t_0=0$  at the origins of the two systems when they are passing each other. Assuming these conditions, together with the hypothesis that the velocity of light has a constant value relative to all inertial systems, we obtain<sup>1</sup> the familiar Lorentz transformation equations:

$$x_0 = (x - vt)(1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \quad ; \quad y = y_0 \quad \text{and} \quad z = z_0 \quad .$$

Let us now consider a rod such as a meter bar at rest in system  $S_0$  with its axis parallel to the  $X_0$  axis. Designate its two ends by the subscripts  $a$  and  $b$ .

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<sup>1</sup> Richtmyer F. K. Introduction to Modern Physics p. 714 (1934) McGraw Hill Book Co.

# 1. INTRODUCTION

Consider two inertial systems  $S$  and  $S'$ ,  $S'$  being fixed relative to the other and  $S$  being fixed relative to the earth but moving relative to  $S'$ . Throughout this work all angles and distances measured in the latter moving system will be designated by the subscript  $s$ , whereas all angles and distances not bearing the subscript zero will refer to the former fixed system.

Let  $XYZ$  represent a rectangular system of coordinates fixed in the system  $S$  and  $X'Y'Z'$  a parallel set fixed in the system  $S'$ . Let the two systems be oriented so that the velocity  $v$  of  $S'$  relative to  $S$  is in the positive  $X$  direction. Denote the coordinates and time as measured by an observer in  $S$  as  $x, y, z$  and as measured by observer in  $S'$  as  $x', y', z', t'$ . Let the zero of time be determined by taking  $t=t'=0$  at the origin of the two systems when they are passing each other, assuming these conditions together with the hypothesis that the velocity of light has a constant value relative to all inertial systems, we obtain <sup>1</sup> the familiar Lorentz transformation equations:

$$x_0 = (x - vt) \sqrt{1 - \frac{v^2}{c^2}}; \quad y_0 = y; \quad z_0 = z; \quad t_0 = t - \frac{vx}{c^2}$$

Let us now consider a rod such as a meter bar at rest in system  $S'$  with its ends parallel to the  $X'$  axis. Measure the two ends by the microscopes  $a$  and  $b$ .

<sup>1</sup> Einstein, A. E. Introduction to Modern Physics, N. Y. (1923) McGraw Hill Book Co.



Then:

$$X_{o_b} = (x_b - vt)(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$$

and

$$X_{o_a} = (x_a - vt)(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$$

provided that  $x_b$  and  $x_a$  are determined at the same time  $t_o$ .

Subtracting, we obtain

$$X_b - X_a = (x_b - x_a)(1 - \frac{v^2}{c^2})^{\frac{1}{2}} \quad \text{Eqn. (1)}$$

From this last equation we see that the measured length of the rod as determined in system S is less than that as determined in  $S_o$  according to the ratio

$$(1 - \frac{v^2}{c^2})^{\frac{1}{2}} : 1$$

From the equations involving y and z, it may be observed that there is no contraction along the y and z coordinates.

Thus, summing up, we may say that according to the Lorentz transformations there is a contraction along a line parallel to the direction of the velocity, but no contraction along a line perpendicular to the direction of the velocity.

#### Effect of Lorentz Transformation on Angles

Let us now consider the effect of the Lorentz transformation on angles. There are two cases that must be considered. First, angles of which one side is parallel to the direction of the velocity. The angles  $\phi$ ,  $(\theta + \phi)$  and  $(\theta + \theta' + \phi)$  are in this group. We shall refer to such angles as angles of Type I. The second type consists of angles of which neither side is parallel to the direction of the velocity. The angles  $\theta$  and  $\theta'$  are of this type. We shall refer to such angles as angles of Type II.

Then

$$X'_D = (X_D - vI) \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

and

$$X'_S = (X_S - vI) \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

provided that  $X'_D$  and  $X'_S$  are determined at the same time  $t'$ .

Substituting, we obtain

$$X'_D - X'_S = (X_D - X_S) \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \quad \text{Eqn. (1)}$$

From this last equation we see that the measured length of the rod

as determined in system  $S$  is less than that as determined in  $S'$  according to

the ratio

$$1 - \frac{v^2}{c^2}$$

From the equations involving  $y$  and  $z$ , it may be observed that

there is no contraction along the  $y$  and  $z$  coordinates.

Thus, summing up, we may say that according to the Lorentz trans-

formation there is a contraction along a line parallel to the direction of

the velocity, but no contraction along a line perpendicular to the direction

of the velocity.

### Effect of Lorentz Transformation on angles

Let us now consider the effect of the Lorentz transformation on

angles. There are two cases that must be considered. First, angles of

which one side is parallel to the direction of the velocity. The angles

$\theta$ ,  $(90^\circ + \theta)$  and  $(90^\circ - \theta)$  are in this group. We shall refer to such

angles as angles of Type I. The second type consists of angles of which

neither side is parallel to the direction of the velocity. The angles  $\theta$  and

$\theta'$  are of this type. We shall refer to such angles as angles of Type II.



## Angles of Type I

In fig. II, let EA and AC include an angle  $\phi_0$  as measured by observer moving in direction AC or  $\underline{v}$ .

Due to the motion along  $\underline{v}$ , the component of AE parallel to  $\underline{v}$  will undergo a contraction and the angle that will be measured by the observer fixed in the ether will be DAB or  $\phi$ .

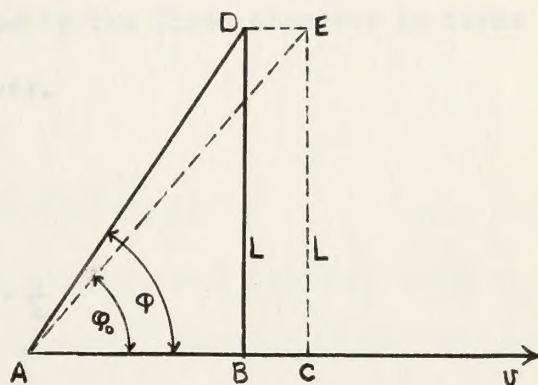


Fig. II

Let L be component of AD perpendicular to  $\underline{v}$

Thus:

$$AC = L \cot \phi_0$$

and from equation (1)

$$AB = L \cot \phi_0 (1 - \frac{v^2}{c^2})^{\frac{1}{2}} .$$

Also

$$\tan \phi = L/AB$$

$$= L / L \cot \phi_0 (1 - \frac{v^2}{c^2})^{\frac{1}{2}}$$

$$= \tan \phi_0 (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} .$$

Expand and drop all terms involving powers of  $v/c$  greater than the second. Throughout this work, we shall not keep higher powers than the second.

Therefore:

$$\tan \phi = \tan \phi_0 (1 + \frac{1}{2} \frac{v^2}{c^2} + \dots) .$$

Eqn. (2)

# Angles of Type I

In Fig. 11, let  $EA$  and  $AB$

include an angle  $\phi$  as measured by

observer moving in direction  $AC$  or  $I$ .

Let the motion along  $x$ , the

component of  $AB$  parallel to  $x$  will

undergo a contraction and the angle

that will be measured by the observer

then in the ether will be  $MA$  or  $\psi$ .

Let  $L$  be component of  $AB$  perpendicular to  $x$

Then:

$$AC = L \sin \phi$$

and from equation (1)

$$AB = L \sin \phi \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

Also

$$\tan \psi = L/AB$$

$$= L / L \sin \phi \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

$$= \tan \phi \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Expand and drop all terms involving powers of  $v/c$  greater than

the second. Throughout this work, we shall not keep higher powers than the

second.

Therefore:

$$\tan \psi = \tan \phi \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \dots\right) \quad \text{Eq. (2)}$$



In the following work we shall deal almost exclusively with sines and cosines. Therefore it will be convenient to derive equations of the sines and cosines of the angles as measured by the fixed observer in terms of angles as measured by the moving observer.

$$\begin{aligned}\cos \varphi &= (1 + \tan^2 \varphi)^{-\frac{1}{2}} \\ &= [1 + \tan^2 \varphi_0 (1 + \frac{v^2}{c^2})]^{-\frac{1}{2}} \\ &= \frac{1}{(\sec^2 \varphi_0 + \frac{v^2}{c^2} \tan^2 \varphi_0)^{\frac{1}{2}}}\end{aligned}$$

Multiply numerator and denominator by  $\cos \varphi_0$ .

$$\cos \varphi = \frac{\cos \varphi_0}{(1 + \frac{v^2}{c^2} \sin^2 \varphi_0)^{\frac{1}{2}}}$$

Expand in terms of powers of  $v/c$ .

$$\cos \varphi = \cos \varphi_0 (1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \varphi_0 + \dots) \quad \text{Eqn. (3)}$$

Also

$$\sin \varphi = \cos \varphi \tan \varphi$$

$$\sin \varphi = \sin \varphi_0 (1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2 \varphi_0 + \dots) \quad \text{Eqn. (4)}$$

Similarly replacing  $\varphi$  by  $(\theta + \varphi)$  and  $\varphi_0$  by  $(\theta_0 + \varphi_0)$  we obtain

$$\sin (\theta + \varphi) = \sin (\theta_0 + \varphi_0) [1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2 (\theta_0 + \varphi_0) + \dots] \quad \text{Eqn. (5)}$$

and

$$\cos (\theta + \varphi) = \cos (\theta_0 + \varphi_0) [1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 (\theta_0 + \varphi_0) + \dots] \quad \text{Eqn. (6)}$$

In the following work we shall deal almost exclusively with sines and cosines. Therefore it will be convenient to derive equations of the sines and cosines of the angles as measured by the fixed observer in terms of angles as measured by the moving observer.

$$\cos \phi = (1 + \tan^2 \phi)^{-\frac{1}{2}}$$

$$= [1 + \tan^2 \phi_0 (1 + \frac{v^2}{c^2})]^{-\frac{1}{2}}$$

$$= \frac{1}{\sec^2 \phi_0 + \frac{v^2}{c^2} \tan^2 \phi_0}$$

Multiply numerator and denominator by  $\cos^2 \phi_0$

$$\cos \phi = \frac{\cos \phi_0}{(1 + \frac{v^2}{c^2} \tan^2 \phi_0)^{\frac{1}{2}}}$$

Expand in terms of powers of  $v/c$ .

$$\cos \phi = \cos \phi_0 (1 - \frac{1}{2} \frac{v^2}{c^2} \tan^2 \phi_0 + \dots) \quad \text{Eqn. (3)}$$

Also

$$\sin \phi = \cos \phi \tan \phi$$

$$\sin \phi = \sin \phi_0 (1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2 \phi_0 + \dots) \quad \text{Eqn. (4)}$$

Similarly replacing  $\phi$  by  $(\phi + \theta)$  and  $\phi_0$  by  $(\phi_0 + \theta_0)$  we obtain

$$\sin (\phi + \theta) = \sin (\phi_0 + \theta_0) [1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2 (\phi_0 + \theta_0) + \dots] \quad \text{Eqn. (5)}$$

and

$$\cos (\phi + \theta) = \cos (\phi_0 + \theta_0) [1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 (\phi_0 + \theta_0) + \dots] \quad \text{Eqn. (6)}$$



Also, replacing  $\phi$  by  $(\theta + \theta' + \phi)$  and  $\phi_0$  by  $(2\theta_0 + \phi_0)$  we get

$$\sin(\theta + \theta' + \phi) = \sin(2\theta_0 + \phi_0) \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2(2\theta_0 + \phi_0) + \dots \right] \quad \text{Eqn. (7)}$$

and

$$\cos(\theta + \theta' + \phi) = \cos(2\theta_0 + \phi_0) \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2(2\theta_0 + \phi_0) + \dots \right] \quad \text{Eqn. (8)}$$

It will be observed that if any of these angles are set equal to 180 degrees, its value undergoes no change upon transformation. However, this statement is not true of angles of 90 degrees.

#### Angles of Type II

To determine the sine and the cosine of angles of Type II in terms of angles measured by an observer in the moving system, let us consider the identity:

$$\theta = (\theta + \phi) - \phi$$

then:

$$\sin \theta = \sin[(\theta + \phi) - \phi]$$

$$= \sin(\theta + \phi) \cos \phi - \cos(\theta + \phi) \sin \phi$$

Substitute equations 3, 4, 5, and 6.

$$\begin{aligned} \sin \theta &= \sin(\theta_0 + \phi_0) \cos \phi_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0) \right] \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \phi_0 \right] \\ &\quad - \cos(\theta_0 + \phi_0) \sin \phi_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2(\theta_0 + \phi_0) \right] \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2 \phi_0 \right] \end{aligned}$$

Multiply and arrange according to powers of  $v/c$ .

$$\begin{aligned} \sin \theta &= \sin(\theta_0 + \phi_0) \cos \phi_0 - \cos(\theta_0 + \phi_0) \sin \phi_0 \\ &\quad + \frac{1}{2} \frac{v^2}{c^2} [\sin(\theta_0 + \phi_0) \cos \phi_0 \cos^2(\theta_0 + \phi_0) - \sin(\theta_0 + \phi_0) \cos \phi_0 \sin^2 \phi_0 \\ &\quad + \cos(\theta_0 + \phi_0) \sin^2(\theta_0 + \phi_0) \sin \phi_0 - \cos(\theta_0 + \phi_0) \sin \phi_0 \cos^2 \phi_0] \end{aligned}$$

Also, replacing  $\theta$  by  $\theta + \phi$  and  $\phi$  by  $\phi + \theta$  we get

$$\sin(\theta + \phi) = \sin(\theta + \phi) \left[ 1 + \frac{v}{c} \cos(\theta + \phi) \right] \quad (1)$$

and

$$\cos(\theta + \phi) = \cos(\theta + \phi) \left[ 1 - \frac{v}{c} \sin(\theta + \phi) \right] \quad (2)$$

It will be observed that in any of these cases the resulting value is the same as the value obtained by the ordinary trigonometric formulae. However, this statement is not true of angles of 90 degrees.

Angles of Type II

To determine the sine and the cosine of angles of Type II in

terms of angles measured by an observer in the moving system, let us

consider the identity:

$$\phi = (\phi + \theta) - \theta$$

$$\sin \phi = \sin[(\phi + \theta) - \theta]$$

$$= \sin(\phi + \theta) \cos \theta - \cos(\phi + \theta) \sin \theta$$

$$\sin \phi = \sin(\phi + \theta) \cos \theta \left[ 1 + \frac{v}{c} \cos(\phi + \theta) \right] - \cos(\phi + \theta) \sin \theta \left[ 1 - \frac{v}{c} \sin(\phi + \theta) \right]$$

$$= \sin(\phi + \theta) \cos \theta \left[ 1 + \frac{v}{c} \cos(\phi + \theta) \right] - \cos(\phi + \theta) \sin \theta \left[ 1 - \frac{v}{c} \sin(\phi + \theta) \right]$$

Multiply and arrange according to powers of  $v/c$ .

$$\sin \phi = \sin(\phi + \theta) \cos \theta - \cos(\phi + \theta) \sin \theta + \frac{v}{c} [\sin(\phi + \theta) \cos \theta \cos(\phi + \theta) - \cos(\phi + \theta) \sin \theta \sin(\phi + \theta)]$$



Replace  $\cos^2(\theta_0 + \phi_0)$  by  $1 - \sin^2(\theta_0 + \phi_0)$  and  $\sin^2\phi_0$  by  $1 - \cos^2\phi_0$ .

Then

$$\begin{aligned}\sin\theta &= \sin\theta_0 + \frac{1}{2} \frac{v^2}{c^2} [\sin(\theta_0 + \phi_0) \cos\phi_0 - \sin^3(\theta_0 + \phi_0) \cos\phi_0 \\ &\quad - \sin(\theta_0 + \phi_0) \cos\phi_0 + \sin(\theta_0 + \phi_0) \cos^3\phi_0 \\ &\quad + \cos(\theta_0 + \phi_0) \sin^2(\theta_0 + \phi_0) \sin\phi_0 - \cos(\theta_0 + \phi_0) \sin\phi_0 \cos^2\phi_0]\end{aligned}$$

$$= \sin\theta_0 + \frac{1}{2} \frac{v^2}{c^2} [-\sin^2(\theta_0 + \phi_0) \sin\theta_0 + \sin\theta_0 \cos^2\phi_0]$$

$$\sin\theta = \sin\theta_0 [1 - \frac{1}{2} \frac{v^2}{c^2} \{\sin^2(\theta_0 + \phi_0) - \cos^2\phi_0\}]$$

Eqn. (9)

Again, let us assume the identity:

$$\theta = (\theta + \phi) - \phi$$

then:

$$\begin{aligned}\cos\theta &= \cos[(\theta + \phi) - \phi] \\ &= \cos(\theta + \phi) \cos\phi + \sin(\theta + \phi) \sin\phi\end{aligned}$$

Substitute equations 3, 4, 5, and 6.

$$\begin{aligned}\cos\theta &= \cos(\theta_0 + \phi_0) \cos\phi_0 [1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2(\theta_0 + \phi_0)] [1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2\phi_0] \\ &\quad + \sin(\theta_0 + \phi_0) \sin\phi_0 [1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0)] [1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2\phi_0]\end{aligned}$$

Multiply and arrange according to powers of  $v/c$ .

$$\begin{aligned}\cos\theta &= \cos(\theta_0 + \phi_0) \cos\phi_0 + \sin(\theta_0 + \phi_0) \sin\phi_0 \\ &\quad + \frac{1}{2} \frac{v^2}{c^2} [-\cos(\theta_0 + \phi_0) \cos\phi_0 \sin^2\phi_0 - \cos(\theta_0 + \phi_0) \sin^2(\theta_0 + \phi_0) \cos\phi_0 \\ &\quad + \sin(\theta_0 + \phi_0) \sin\phi_0 \cos^2\phi_0 + \sin(\theta_0 + \phi_0) \cos^2(\theta_0 + \phi_0) \sin\phi_0]\end{aligned}$$

Replace  $\sin^2(\theta_0 + \phi_0)$  by  $1 - \cos^2(\theta_0 + \phi_0)$  and  $\sin^2\phi_0$  by  $1 - \cos^2\phi_0$ .

replace  $\cos \theta$  ( $\theta = \frac{\pi}{2}$ ) by  $1 - \sin^2 \theta$  and  $\sin \theta$  by  $1 - \cos^2 \theta$ .

Thus

$$\begin{aligned} \sin 2 &= 2 \sin \theta \cos \theta = 2 \sin \theta (1 - \sin^2 \theta) \\ &= 2 \sin \theta - 2 \sin^3 \theta \\ &= 2 \sin \theta - 2 \sin \theta \cos^2 \theta \\ &= 2 \sin \theta (1 - \cos^2 \theta) \end{aligned}$$

$$= 2 \sin \theta \left[ 1 - \frac{1}{2} (1 + \cos 2\theta) \right]$$

$$= 2 \sin \theta \left[ 1 - \frac{1}{2} (1 + \cos 2\theta) \right]$$

Let

Again, let us assume the identity

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

Then

$$\begin{aligned} \cos 2 &= \cos(\theta + \phi) \\ &= \cos \theta \cos \phi - \sin \theta \sin \phi \end{aligned}$$

Substitute  $\theta = \frac{\pi}{2}$  and  $\phi = \theta$ .

$$\begin{aligned} \cos \theta &= \cos(\theta + \theta) \\ &= \cos \theta \cos \theta - \sin \theta \sin \theta \\ &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

Multiply and arrange according to powers of  $\theta$ .

$$\begin{aligned} \cos 2 &= \cos(\theta + \theta) \\ &= \cos \theta \cos \theta - \sin \theta \sin \theta \\ &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

Replace  $\sin^2 \theta$  ( $\theta = \frac{\pi}{2}$ ) by  $1 - \cos^2 \theta$  and  $\sin \theta$  by  $1 - \cos \theta$ .



Then

$$\begin{aligned}\cos \vartheta &= \cos \vartheta_0 + \frac{1}{2} \frac{v^2}{c^2} [-\cos(\vartheta_0 + \varphi_0) \cos \varphi_0 + \cos(\vartheta_0 + \varphi_0) \cos^3 \varphi_0 - \cos(\vartheta_0 + \varphi_0) \cos \varphi_0 \\ &\quad + \cos^3(\vartheta_0 + \varphi_0) \cos \varphi_0 + \sin(\vartheta_0 + \varphi_0) \sin \varphi_0 \cos^2 \varphi_0 \\ &\quad + \sin(\vartheta_0 + \varphi_0) \cos^2(\vartheta_0 + \varphi_0) \sin \varphi_0]\end{aligned}$$

$$= \cos \vartheta_0 + \frac{1}{2} \frac{v^2}{c^2} [\cos^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \cos^2 \varphi_0 \cos \vartheta_0 - 2 \cos(\vartheta_0 + \varphi_0) \cos \varphi_0]$$

$$\begin{aligned}= \cos \vartheta_0 + \frac{1}{2} \frac{v^2}{c^2} [\cos^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \cos \varphi_0 \{ \cos \varphi_0 \cos \vartheta_0 - 2 \cos \vartheta_0 \cos \varphi_0 \\ + 2 \sin \vartheta_0 \sin \varphi_0 \}]\end{aligned}$$

$$\cos \vartheta = \cos \vartheta_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2(\vartheta_0 + \varphi_0) + \cos \varphi_0 (2 \tan \vartheta_0 \sin \varphi_0 - \cos \varphi_0) \} + \dots \right] \quad \text{Eqn. (10)}$$

Similarly, to obtain equations for  $\theta'$  replace  $\phi_0$  by  $(\theta_0 + \phi_0)$ .

This yields

$$\sin \vartheta' = \sin \vartheta_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \sin^2(2\vartheta_0 + \varphi_0) - \cos^2(\vartheta_0 + \varphi_0) \} + \dots \right] \quad \text{Eqn. (11)}$$

and

$$\cos \vartheta' = \cos \vartheta_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2(2\vartheta_0 + \varphi_0) + \cos(\vartheta_0 + \varphi_0) [2 \tan \vartheta_0 \sin(\vartheta_0 + \varphi_0) - \cos(\vartheta_0 + \varphi_0)] \} + \dots \right] \quad \text{Eqn. (12)}$$

In later work, we shall find that the angle  $(\theta - \theta')$  occurs frequently, therefore it will be convenient to derive the value of  $\sin(\theta - \theta')$  and  $\cos(\theta - \theta')$  at this point.

$$\sin(\vartheta - \vartheta') = \sin \vartheta \cos \vartheta' - \cos \vartheta \sin \vartheta'$$

Substitute equations 9, 10, 11, and 12.

$$\begin{aligned}\sin(\vartheta - \vartheta') &= \sin \vartheta_0 \cos \vartheta_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \sin^2(\vartheta_0 + \varphi_0) - \cos^2 \varphi_0 - \cos^2(2\vartheta_0 + \varphi_0) - 2 \tan \vartheta_0 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) + \cos^2(\vartheta_0 + \varphi_0) \} \right] \\ &\quad - \sin \vartheta_0 \cos \vartheta_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2(\vartheta_0 + \varphi_0) + 2 \tan \vartheta_0 \sin \varphi_0 \cos \varphi_0 - \cos^2 \varphi_0 - \sin^2(2\vartheta_0 + \varphi_0) + \cos^2(\vartheta_0 + \varphi_0) \} \right]\end{aligned}$$

$$\begin{aligned} \cos \theta &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi \\ &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi \end{aligned}$$

$$\cos \theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi$$

$$\cos \theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi$$

$$\cos \theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi \quad (10)$$

Similarly, we obtain equations for  $\theta'$ , replace  $\theta$  by  $\theta'$ .

Then we have

$$\cos \theta' = \cos \theta_1' \cos \theta_2' + \sin \theta_1' \sin \theta_2' \cos \phi' \quad (11)$$

and

$$\cos \theta = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi \quad (12)$$

In later parts, we shall find that the angle  $\theta = \theta'$  occurs frequently, therefore it will be convenient to denote the value of  $\sin \theta$  by  $\sin \theta$  and  $\cos \theta$  by  $\cos \theta$ .

$$\sin \theta = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$$

Substitute equations 10, 11, and 12.

$$\begin{aligned} \sin \theta &= \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \\ &= \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \end{aligned}$$



Multiply and arrange according to powers of  $v/c$ .

$$\sin(\theta - \theta') = \frac{1}{2} \frac{v^2}{c^2} \sin \theta_0 \cos \theta_0 \{ -\sin^2(\theta_0 + \phi_0) + 2 \cos^2 \phi_0 + \cos^2(2\theta_0 + \phi_0) - 2 \cos^2(\theta_0 + \phi_0) + \sin^2(2\theta_0 + \phi_0) - \cos^2(\theta_0 + \phi_0) + 2 \tan \theta_0 [\sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) - \sin \phi_0 \cos \phi_0] \}$$

$$\sin(\theta - \theta') = \frac{v^2}{c^2} \sin \theta_0 \cos \theta_0 \{ \cos^2 \phi_0 - \cos^2(\theta_0 + \phi_0) + \tan \theta_0 [\sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) - \sin \phi_0 \cos \phi_0] \} + \dots \quad \text{Eqn. (13)}$$

Also

$$\cos(\theta - \theta') = [1 - \sin^2(\theta - \theta')]^{\frac{1}{2}}$$

$$\cos(\theta - \theta') = 1 + \dots \quad \text{Eqn. (14)}$$

neglecting terms involving powers of  $v/c$  greater than the second.

It will be desirable now to determine the value of the sine and the cosine of angle  $\sigma$ . The nature of the angle  $\sigma$  was discussed on page 15.

Let fig. IIIa represent  $M'$  as seen by the observer fixed in the ether and fig. IIIb,  $M'$  as seen by the moving observer. It is clear that

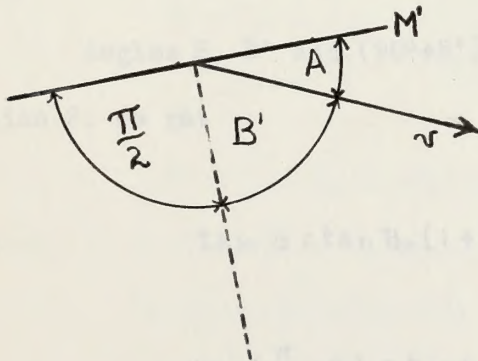


Fig. IIIa

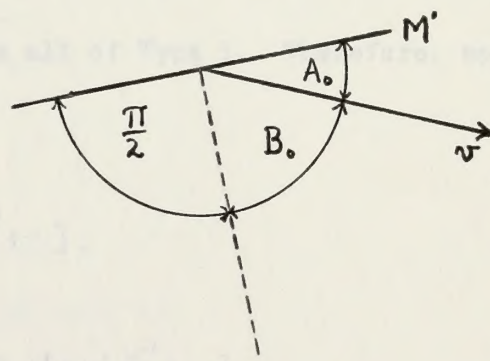


Fig. IIIb

due to the contraction of the mirror in the direction of  $\underline{v}$  the mirror suffers a slight rotation and therefore the normals constructed to the mirror by the two observers will be different. The angle between these two normals is called  $\sigma$ .

In fig. IIIa the dotted line represents the normal to the mirror

multiply and arrange according to powers of  $v/c$ .

$$\cos(\theta - \theta') = \frac{1}{2} \left( \cos \theta + \cos \theta' + \frac{v}{c} (\sin \theta + \sin \theta') + \frac{v^2}{c^2} (\cos \theta + \cos \theta') + \dots \right)$$

$$\cos(\theta - \theta') = \frac{1}{2} \left( \cos \theta + \cos \theta' + \frac{v}{c} (\sin \theta + \sin \theta') + \frac{v^2}{c^2} (\cos \theta + \cos \theta') + \dots \right)$$

$$\cos(\theta - \theta') = [1 - \frac{v^2}{c^2} \sin^2 \theta]^{1/2}$$

$$\cos(\theta - \theta') = 1 + \dots$$

neglecting terms involving powers of  $v/c$  greater than the second.

It will be recalled that in determining the value of the sine and cosine of angles, the nature of the angle was discussed on page 10. At this point we can see by the observer fixed in the ether and Fig. 11a, as seen by the moving observer. It is clear that



Fig. 11a Fig. 11b

due to the contraction of the ether in the direction of  $y$  the mirror reflects a right rotation and therefore the normal is reflected in the ether by the two observers with no difference. The angle between these two normals is called  $\theta$ . In Fig. 11a the dotted line represents the normal to the mirror



constructed by the fixed observer. In his coordinate system  $A+B'=90^\circ$ .

Similarly in fig. IIIb the dotted line represents the normal to the mirror

constructed by the moving observer. In his coordinate system  $A_0+B_0=90^\circ$ .

Referring to figure I, we can deduce that

$$B_0 = (\pi - 2\theta_0 - \varphi_0).$$

The angle  $B_0$  becomes angle  $B'$  in the fixed observer's coordinate system.

Thus

$$B' = (\pi - \theta - \theta' - \varphi).$$

However, in figure I we have designated this same angle as

$$B = (\pi - \theta - \theta' - \varphi - \sigma).$$

Therefore, we have

$$B' = B + \sigma.$$

Angles  $B$ ,  $B'$  and  $(90^\circ + B')$  are all of Type I. Therefore, applying equation 2, we get

$$\tan B = \tan B_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right].$$

Also

$$\tan \left( \frac{\pi}{2} + B' \right) = \tan \left( \frac{\pi}{2} + B_0 \right) \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right]$$

or

$$\operatorname{ctn} B' = \operatorname{ctn} B_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right].$$

Since

$$B' = B + \sigma$$

$$\operatorname{ctn} B' = \operatorname{ctn}(B + \sigma) = \operatorname{ctn} B_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right].$$

Let us expand  $\operatorname{ctn}(B + \sigma)$ .

constituted by the fixed observer. In the coordinate system  $A_1, A_2, A_3$ , the velocity is  $u$ . The line joining the observer to the moving observer is the normal to the surface constituted by the moving observer. In the coordinate system  $A_1, A_2, A_3$ , the velocity is  $u$ . Referring to figure 1, we can deduce that

$$\theta' = (\pi - \theta) - \alpha,$$

The angle  $\theta'$  becomes angle  $\theta''$  in the fixed observer's coordinate system.

$$\theta'' = (\pi - \theta' - \alpha) - \alpha,$$

However, in figure 1 we have determined this same angle as

$$\theta = (\pi - \theta' - \alpha) - \alpha,$$

Therefore, we have

$$\theta' = \theta + \alpha.$$

Angles  $\theta'$ ,  $\theta''$ , and  $(\theta + \alpha)$  are all of type I. Therefore, applying

equation (2), we get

$$\tan \theta' = \tan(\theta + \alpha) = \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha}.$$

Also

$$\tan(\theta' + \alpha) = \tan(\theta + 2\alpha) = \frac{\tan(\theta + \alpha) + \tan \alpha}{1 - \tan(\theta + \alpha) \tan \alpha}.$$

or

$$\tan \theta'' = \tan(\theta' + \alpha) = \frac{\tan \theta' + \tan \alpha}{1 - \tan \theta' \tan \alpha}.$$

Since

$$\theta' = \theta + \alpha,$$

$$\tan \theta'' = \tan(\theta + 2\alpha) = \frac{\tan \theta + \tan 2\alpha}{1 - \tan \theta \tan 2\alpha}.$$

Let us expand  $\tan 2\alpha$



$$\frac{\text{ctn } B \text{ ctn } \sigma - 1}{\text{ctn } \sigma + \text{ctn } B} = \text{ctn } B_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right]$$

Now,

$$\text{ctn } B = \frac{1}{\tan B} = \text{ctn } B_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} + \dots \right]$$

Substitute the latter expression in the equation preceeding it.

$$\frac{\text{ctn } B_0 \text{ ctn } \sigma \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \right] - 1}{\text{ctn } \sigma + \text{ctn } B_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \right]} = \text{ctn } B_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \right]$$

Cross multiply:

$$\text{ctn } B_0 \text{ ctn } \sigma - \frac{1}{2} \frac{v^2}{c^2} \text{ctn } B_0 \text{ ctn } \sigma - 1 = \text{ctn } B_0 \text{ ctn } \sigma + \frac{1}{2} \frac{v^2}{c^2} \text{ctn } B_0 \text{ ctn } \sigma + \text{ctn}^2 B_0$$

From which we obtain

$$\text{ctn } \sigma = - \frac{1}{\frac{v^2}{c^2} \sin^2 B_0 \text{ctn } B_0}$$

and

$$\tan \sigma = - \frac{v^2}{c^2} \sin B_0 \cos B_0$$

Substitute for  $B_0$  its value  $(\pi - 2\theta_0 - \phi_0)$  then,

$$\tan \sigma = - \frac{v^2}{c^2} \sin(\pi - 2\theta_0 - \phi_0) \cos(\pi - 2\theta_0 - \phi_0)$$

$$\tan \sigma = \frac{v^2}{c^2} \sin(2\theta_0 + \phi_0) \cos(2\theta_0 + \phi_0).$$

Also

$$\cos \sigma = (1 + \tan^2 \sigma)^{-\frac{1}{2}} = 1 + \dots \quad \text{Eqn. (15)}$$

neglecting powers of  $v/c$  greater than the second.

and

$$\sin \sigma = \cos \sigma \tan \sigma = \frac{v^2}{c^2} \sin(2\theta_0 + \phi_0) \cos(2\theta_0 + \phi_0) \quad \text{Eqn. (16)}$$

$$\frac{c \sin \delta \sin \alpha - 1}{c \sin \alpha - c \sin \delta} = \frac{1 - \frac{1}{2} \frac{v^2}{c^2}}{1 - \frac{1}{2} \frac{v^2}{c^2}}$$

or,

$$c \sin \delta = \frac{1}{\cos \delta} = \frac{1}{\cos \delta} \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \right) \left( 1 + \frac{1}{2} \frac{v^2}{c^2} \right)$$

Substitute the latter expression in the equation preceding it.

$$\frac{1 + \frac{1}{2} \frac{v^2}{c^2} \sin^2 \delta - \frac{1}{2} \frac{v^2}{c^2}}{c \sin \alpha - c \sin \delta} = \frac{1 - \frac{1}{2} \frac{v^2}{c^2}}{1 - \frac{1}{2} \frac{v^2}{c^2}}$$

Cross multiply:

$$c \sin \delta \left( c \sin \alpha - \frac{1}{2} \frac{v^2}{c^2} c \sin \delta \right) = c \sin \delta \left( c \sin \alpha - \frac{1}{2} \frac{v^2}{c^2} c \sin \delta \right) + \frac{1}{2} \frac{v^2}{c^2} c \sin \delta \left( c \sin \alpha - \frac{1}{2} \frac{v^2}{c^2} c \sin \delta \right)$$

From which we obtain

$$c \sin \alpha = \frac{1}{\cos \delta} = \frac{1}{\cos \delta} \left( 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \delta \right)$$

and

$$\tan \alpha = -\frac{v}{c} \sin \delta \cos \delta$$

Substitute for  $\alpha$  the value  $(\pi - \delta - \delta_0)$  then,

$$\tan \alpha = -\frac{v}{c} \sin \delta \cos \delta = -\frac{v}{c} \sin \delta \cos \delta$$

$$\tan \alpha = -\frac{v}{c} \sin \delta \cos \delta$$

Also

$$\cos \alpha = (1 + \tan^2 \alpha)^{-\frac{1}{2}} = 1 \quad (10)$$

neglecting powers of  $v/c$  greater than the second.

and

$$\sin \alpha = \cos \delta \tan \alpha = \frac{v}{c} \sin \delta \cos \delta \cos \delta \quad (11)$$



We shall now determine the sine and cosine of angle  $\omega$ . The nature of this angle was discussed on page 15 .

Let figure IVa represent  $M''$  as seen by the observer fixed in the ether and figure IVb,  $M''$  as seen by the moving observer. As in the case of

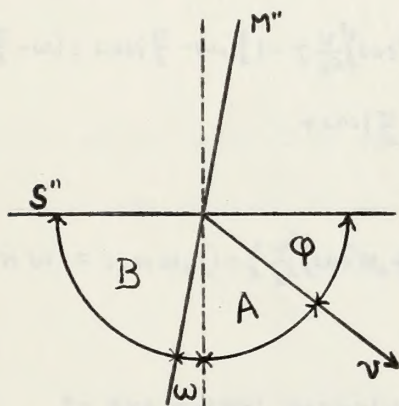


Fig. IVa

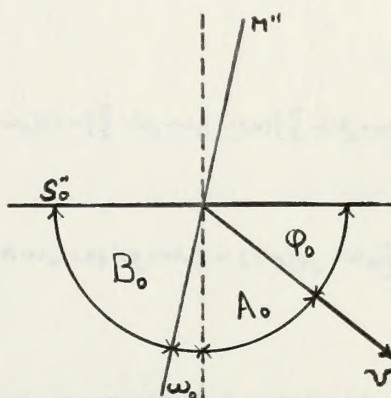


Fig. IVb

the mirror  $M'$ , the mirror  $M''$  undergoes a contraction in the direction of  $\underline{v}$ . The effect of this contraction is that the mirror is slightly rotated.

As was explained on page 15, a small angle must be introduced at  $M''$  so that the fringes observed in the telescope may be adjusted to some convenient width. Let  $\omega$  represent the angle introduced in the adjustment of the fringes as measured by the observer fixed in the ether, plus the small angle which comes into existence from the change of orientation of the mirror caused by the contraction of  $M''$ .

In figure IVa , the dotted line represents the normal to  $S''$  constructed by the observer fixed in the ether. In his coordinate system  $\omega = 90^\circ - \beta$ .

In figure IVb, the dotted line represents the normal to  $S''_0$  constructed by the moving observer. In his coordinate system,  $\omega_0 = 90^\circ - \beta_0$  .

Also  $A_0 = 90^\circ - \phi_0 + \omega_0$  and  $B_0 = 90^\circ - \omega_0$  .

We shall now determine the size and cosine of angles. The nature of this angle was discussed on page 15. Let figure IV represent  $M'$  as seen by the observer from the ether and figure IV' as seen by the moving observer. As in the case of



Fig. IV



Fig. IV'

the mirror  $M'$ , the mirror  $M$  undergoes a contraction in the direction of  $y$ . The effect of this contraction is that the mirror is slightly rotated. As was explained on page 15, a small angle must be introduced at  $M'$  so that the triangle observed in the telescope may be adjusted to same convenient width. Lines represent the angle introduced in the adjustment of the triangle as measured by the observer from the ether, also the small angle which comes into existence from the change of orientation of the mirror caused by the contraction of  $M'$ .

In figure IV, the dotted line represents the normal to  $S'$  contracted by the observer from the ether. In his coordinate system  $W-V'$  in figure IV', the dotted line represents the normal to  $S'$  contracted by the moving observer. In his coordinate system  $W'-V'$  also  $\psi' = \psi - \phi$  and  $\psi = \psi' + \phi$ .



By equation 10

$$\cos B = \cos B_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2(A_0 + B_0) + \cos A_0 [2 \tan B_0 \sin A_0 - \cos A_0] \} + \dots \right]$$

Substitute values of B, B<sub>0</sub> and A<sub>0</sub>.

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \omega\right) &= \cos\left(\frac{\pi}{2} - \omega_0\right) \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2(\pi - \varphi_0) \right. \\ &\quad \left. + \cos\left\{\frac{\pi}{2} - (\varphi_0 - \omega_0)\right\} [2 \tan\left(\frac{\pi}{2} - \omega_0\right) \sin\left\{\frac{\pi}{2} - (\varphi_0 - \omega_0)\right\} - \cos\left\{\frac{\pi}{2} - (\varphi_0 - \omega_0)\right\}] \} + \dots \right] \end{aligned}$$

Or

$$\sin \omega = \sin \omega_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2 \varphi_0 + \sin(\varphi_0 - \omega_0) \{ 2 \tan \omega_0 \cos(\varphi_0 - \omega_0) - \sin(\varphi_0 - \omega_0) \} \} + \dots \right],$$

In the actual experiment it is found convenient to make  $\omega_0$  of the order of magnitude of  $v/c$ . Therefore, we may set

$$\sin \omega_0 = \frac{v}{c} \sin \mu_0$$

where  $\mu_0$  is defined in terms of this equation.

$$\begin{aligned} \cos \omega_0 &= (1 - \sin^2 \omega_0)^{\frac{1}{2}} \\ &= 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \mu_0 + \dots \end{aligned}$$

In equation for  $\sin \omega$  we observe that  $\sin(\varphi_0 - \omega_0)$ ,  $\cos(\varphi_0 - \omega_0)$  and  $\tan \omega_0$  occur in the coefficient of  $(v/c)^2$ . Therefore, it will be necessary to expand them only up to  $(v/c)^{-1}$  after substituting for  $\sin \omega_0$ . Thus, all that remains in the order term is

$$\tan \omega_0 = \frac{\cos \omega_0}{\sin \omega_0} = \frac{1}{\frac{v}{c} \sin \mu_0} + \dots$$

Substituting this value, we get

$$\sin \omega = \frac{v}{c} \sin \mu_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \sin \varphi_0 \left\{ \frac{2 \cos \varphi_0}{\frac{v}{c} \sin \mu_0} \right\} + \dots \right]$$





Or:

$$\sin \omega = \frac{v}{c} \sin \mu_0 + \frac{v^2}{c^2} \sin \varphi_0 \cos \varphi_0 + \dots, \quad \text{Eqn. (17)}$$

And:

$$\begin{aligned} \cos \omega &= (1 - \sin^2 \omega)^{\frac{1}{2}} \\ &= 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \mu_0 + \dots \end{aligned} \quad \text{Eqn. (18)}$$

It is clear from equation 17 that the  $v/c$  term is the part of the angle which was introduced in the adjustment of the mirror; whereas the  $(v/c)^2$  term is the part introduced because of the contraction of the mirror.

#### Effect of Lorentz Transformation on Distances

Referring to fig. I we observe that the angle between  $S'$  and the direction of the velocity is  $(\theta + \theta' + \phi)$ . Therefore the component of the velocity parallel to  $S'$  is  $v \cos(\theta + \theta' + \phi)$ . Substituting this value of  $v$  in equation 1 gives:

$$S' = S'_0 \left[ 1 - \frac{v^2}{c^2} \cos^2(\theta + \theta' + \phi) + \dots \right]^{\frac{1}{2}}$$

Substitute equation 8 and expand.

$$S' = S'_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \cos^2(2\theta_0 + \varphi_0) + \dots \right] \quad \text{Eqn. (19)}$$

Similarly, referring to fig. I we observe that the angle between  $S''$  and the direction of the velocity is  $\phi$ . Therefore, the component of the velocity parallel to  $S''$  is  $v \cos \phi$ . Substituting this value of  $v$  in equation 1 gives:

$$S'' = S''_0 \left[ 1 - \frac{v^2}{c^2} \cos^2 \phi + \dots \right]^{\frac{1}{2}}$$

Substitute equation 3 and expand.

$$S'' = S''_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \cos^2 \varphi_0 + \dots \right] \quad \text{Eqn. (20)}$$

$$(17) \quad \sin \theta = \frac{v}{c} \sin \theta' + \frac{v}{c} \cos \theta' \cos \phi$$

and

$$\cos \theta = (1 - \frac{v^2}{c^2})^{1/2} \cos \theta'$$

$$(18) \quad \sin \theta' = \frac{v}{c} \sin \theta + \frac{v}{c} \cos \theta \cos \phi$$

It is clear from equation (17) that the  $v/c$  term is the part of the

angle which was introduced in the adjustment of the mirror, whereas the  $(v/c)^2$  term is the part introduced because of the contraction of the mirror.

#### Effect of Lorentz Transformation on Distances

Referring to Fig. 1 we observe that the angle between  $S'$  and the

direction of the velocity is  $\theta' = \theta'(\phi)$ . Therefore the component of the velocity parallel to  $S'$  is  $v \cos \theta'(\phi)$ . Substituting this value of  $v$  in

equation (1) gives

$$S' = S \left[ 1 - \frac{v}{c} \cos \theta'(\phi) + \frac{v^2}{c^2} \right]$$

Substituting equation (2) and rearranging

$$S = S' \left[ 1 - \frac{v}{c} \cos \theta'(\phi) + \frac{v^2}{c^2} \right]^{-1}$$

Eq. (19)

Similarly, referring to Fig. 1 we observe that the angle between

$S''$  and the direction of the velocity is  $\theta'' = \theta''(\phi)$ . Therefore, the component of the velocity parallel to  $S''$  is  $v \cos \theta''(\phi)$ . Substituting this value of  $v$  in

equation (1) gives

$$S'' = S' \left[ 1 - \frac{v}{c} \cos \theta''(\phi) + \frac{v^2}{c^2} \right]$$

Substituting equation (2) and rearranging

$$S = S'' \left[ 1 - \frac{v}{c} \cos \theta''(\phi) + \frac{v^2}{c^2} \right]^{-1}$$

Eq. (20)









traverses an equal distance  $CF=ct$ . Construct  $EF$  and  $CF$  so that they include a right angle.  $EF$  represents the wave front of the reflected ray. Let  $i'$  be the angle of reflection, and  $\alpha$  the angle between  $CE$  and the mirror.

The right triangles  $CFE$  and  $CDE$  are congruent since they have a common hypotenuse and equal legs  $CF$  and  $ED$ .

Therefore  $\angle i - \alpha = \angle i' + \alpha$

and  $\alpha = (i - i')/2$

In triangle  $CEG$   $ut = CE \sin \alpha$

In triangle  $CFE$   $ct = CE \sin (i' + \alpha)$

Divide the next to the last equation by the last.

$$\frac{u}{c} = \frac{\sin \alpha}{\sin (i' + \alpha)} = \frac{\sin \frac{(i - i')}{2}}{\sin \frac{(i + i')}{2}}$$

or,  $\sin \frac{(i - i')}{2} = \frac{u}{c} \sin \frac{(i + i')}{2}$

Expanding this expression, we get:

$$\sin \frac{i}{2} \cos \frac{i'}{2} - \cos \frac{i}{2} \sin \frac{i'}{2} = \frac{u}{c} (\sin \frac{i}{2} \cos \frac{i'}{2} + \cos \frac{i}{2} \sin \frac{i'}{2})$$

Divide the last equation by  $\cos \frac{i}{2} \cos \frac{i'}{2}$ .

$$\tan \frac{i}{2} - \tan \frac{i'}{2} = \frac{u}{c} (\tan \frac{i}{2} + \tan \frac{i'}{2})$$

Using Peirce #578  $\left[ \tan \frac{i}{2} = \frac{\sin i}{1 + \cos i} \right]$ , we obtain

$$\frac{\sin i}{1 + \cos i} - \frac{\sin i'}{1 + \cos i'} = \frac{u}{c} \frac{\sin i}{1 + \cos i} + \frac{u}{c} \frac{\sin i'}{1 + \cos i'}$$

Collect terms and replace  $\cos i'$  by  $(1 - \sin^2 i')^{\frac{1}{2}}$ .

$$\frac{\sin i'}{1 + (1 - \sin^2 i')^{\frac{1}{2}}} = A \frac{\sin i}{1 - \cos i}$$

Where  $A$  is defined as  $(c-u)/(c+u)$ .

Cross multiply and rearrange the last expression.

$$\sin i' + \sin i \cos i' - A \sin i = A \sin i (1 - \sin^2 i')^{\frac{1}{2}}$$

Therefore an equal distance  $CE'$ . Connect  $E'$  and  $E$  so that they include a right angle.  $E'$  represents the wave front of the reflected ray. And  $E'$  is the angle of reflection, and is the angle between  $CE$  and the mirror. The right triangles  $CE$  and  $CE'$  are congruent since they have a

common hypotenuse and equal legs  $CE$  and  $CE'$ .

Therefore  $\angle E' = \angle E = \alpha$

and  $\alpha = (i - r)/2$

In triangle  $CE'$   $CE' = CE \sin \alpha$

In triangle  $CE$   $CE = CE \sin (i + \alpha)$

Divide the next to the last equation by the last.

$$\frac{CE'}{CE} = \frac{\sin \alpha}{\sin (i + \alpha)} = \frac{\sin \frac{i - r}{2}}{\sin \frac{i + r}{2}}$$

$$\text{or, } \sin \frac{(i - r)}{2} = \frac{\sin \frac{i + r}{2}}{\sin \frac{i + r}{2}}$$

Expanding this expression, we get:

$$\sin \frac{i}{2} \cos \frac{r}{2} - \cos \frac{i}{2} \sin \frac{r}{2} = \frac{\sin \frac{i}{2} \cos \frac{r}{2} + \cos \frac{i}{2} \sin \frac{r}{2}}{2}$$

Divide the last equation by  $\sin \frac{i}{2} \cos \frac{r}{2}$ .

$$\tan \frac{i}{2} = \tan \frac{r}{2} = \frac{\sin \frac{r}{2}}{\cos \frac{r}{2}}$$

Being given  $\tan \frac{i}{2} = \frac{\sin i}{1 + \cos i}$  we obtain

$$\frac{\sin i}{1 + \cos i} = \frac{\sin r}{1 + \cos r} = \frac{\sin r}{1 + \cos r} = \frac{\sin r}{1 + \cos r} + \frac{\sin r}{1 + \cos r}$$

Collect terms and replace  $\cos i$  by  $(1 - \sin^2 i)^{1/2}$ .

$$\frac{\sin i}{1 + (1 - \sin^2 i)^{1/2}} = A = \frac{\sin r}{1 + \cos r}$$

Given  $A$  is defined as  $(-1 - \sin^2 i)^{1/2}$ .

Cross multiply and rearrange the last expression.

$$\sin i + \sin i \cos i = A \sin r + A \sin r \cos r$$



Squaring both sides yields:

$$\sin i' + \sin i' \cos^2 i + 2 \sin i' \cos i - 2 A \sin i - 2 \sin i \cos i + A^2 \sin^2 i = 0$$

From which we obtain

$$\sin i' = \frac{2 A \sin i}{1 + \cos i + A^2 (1 - \cos i)}$$

Substitute for A, it's value and multiply through by  $(c+u)$ .

$$\sin i' = \frac{2(c^2 - u^2) \sin i}{(c+u)^2 (1 + \cos i) + (c-u)^2 (1 - \cos i)}$$

Expand and divide through by  $c^2$ .

$$\sin i' = \frac{(1 - \frac{u^2}{c^2}) \sin i}{1 + 2 \frac{u}{c} \cos i + \frac{u^2}{c^2}}$$

Divide numerator by denominator and neglect powers of  $u/c$  greater than the second. This yields:

$$\sin i' = \sin i - 2 \frac{u}{c} \sin i \cos i + 2 \frac{u^2}{c^2} \sin i (2 \cos^2 i - 1) + \dots$$

Using the relationship  $\cos i' = (1 - \sin^2 i')^{\frac{1}{2}}$ , we obtain

$$\cos i' = \cos i + 2 \frac{u}{c} \sin^2 i - 4 \frac{u^2}{c^2} \sin^2 i \cos i + \dots$$

Replacing  $u$  by its equal  $v \sin \phi$ , we obtain as the final equations for  $\sin i'$  and  $\cos i'$ :

$$\sin i' = \sin i - 2 \frac{v}{c} \sin \phi \sin i \cos i + 2 \frac{v^2}{c^2} \sin^2 \phi \sin i (2 \cos^2 i - 1) \text{ Eqn. (21)}$$

$$\cos i' = \cos i + 2 \frac{v}{c} \sin \phi \sin^2 i - 4 \frac{v^2}{c^2} \sin^2 \phi \sin^2 i \cos i + \dots \text{ Eqn. (22)}$$

We have thus expressed both the sine and the cosine of the angle of reflection in terms of the sine and the cosine of the angle of incidence;

Expanding both sides yields:

$$\sin \theta' + \sin \theta' \cos \theta + 2 \sin \theta' \cos \theta = 2 \sin \theta' \cos \theta + \sin \theta' \cos \theta + \sin \theta' \cos \theta$$

From which we obtain

$$\sin \theta' = \frac{2 \sin \theta' \cos \theta}{1 + \cos \theta + \sin^2 \theta}$$

Substituting for  $\theta'$  in  $\theta'$  and multiplying through by  $(1 + \cos \theta)$

$$\sin \theta' = \frac{2 \sin \theta' \cos \theta}{(1 + \cos \theta) + (1 - \cos \theta)}$$

Expanding and dividing through by  $\sin \theta'$

$$\sin \theta' = \frac{2 \cos \theta}{1 + \cos \theta + 1 - \cos \theta}$$

Divide numerator by denominator and neglect powers of  $\theta'$  greater than the

second. This yields:

$$\sin \theta' = \sin \theta + \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \sin \theta \cos \theta - \frac{1}{2} \sin \theta \cos \theta$$

Using the relationship  $\cos \theta' = 1 - \sin^2 \theta'$ , we obtain

$$\cos \theta' = \cos \theta + \frac{1}{2} \cos \theta \cos \theta - \frac{1}{2} \cos \theta \cos \theta + \frac{1}{2} \cos \theta \cos \theta$$

Expanding  $\cos \theta'$  by the square of  $\sin \theta'$ , we obtain as the final equation for

$\sin \theta'$  and  $\cos \theta'$ :

$$\sin \theta' = \sin \theta + \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \sin \theta \cos \theta - \frac{1}{2} \sin \theta \cos \theta \quad (21)$$

$$\cos \theta' = \cos \theta + \frac{1}{2} \cos \theta \cos \theta - \frac{1}{2} \cos \theta \cos \theta + \frac{1}{2} \cos \theta \cos \theta \quad (22)$$

We have thus expressed both  $\sin \theta'$  and  $\cos \theta'$  in terms of the angle

of reflection in terms of the angle of the angle of incidence.



the sine of the angle between the mirror and the direction of the velocity; the velocity of light and the velocity of the mirror, this latter velocity being positive when the mirror is approaching the source of light and negative when receding from the source of light.

The first of the three is the first and the second of the three  
the second of the three is the second and the third of the three  
the third of the three is the third and the fourth of the three  
the fourth of the three is the fourth and the fifth of the three  
the fifth of the three is the fifth and the sixth of the three

THE  
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### CHANGE OF WAVE LENGTH UPON REFLECTION FROM A MOVING MIRROR

In figure VI, CO represents a ray of light incident upon the mirror M at O. The lines drawn perpendicular to this ray represent successive wave fronts one wave length apart. Similarly, the lines drawn perpendicular to the reflected ray OD represent reflected wave fronts one wave length apart.

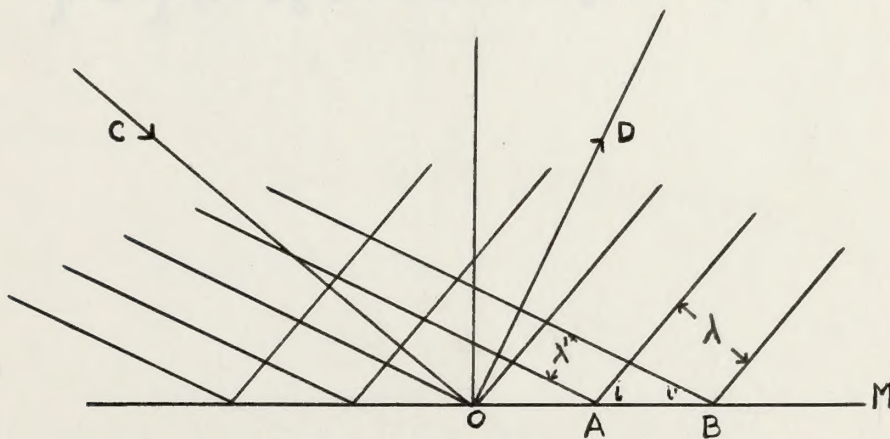


Fig. VI

If  $i$  is the angle of incidence,  $i'$  the angle of reflection,  $\lambda$  the wave length of the incident light and  $\lambda'$  the wave length of the reflected light, we have:

$$\lambda' = AB \sin i'$$

and

$$\lambda = AB \sin i.$$

# THEORY OF THE RAY OF A REFLECTING SURFACE

In figure 1, the ray of light is shown as a straight line. The line from the source to the point of reflection is labeled  $r$ . The line from the point of reflection to the observer is labeled  $R$ . The angle between the ray and the normal is labeled  $i$ . The angle between the reflected ray and the normal is labeled  $r$ . The angle between the incident ray and the reflected ray is labeled  $2i$ .



It is the angle of incidence,  $i$ , the angle of reflection,  $r$ , and the angle between the incident ray and the reflected ray,  $2i$ , that are the angles of reflection.

$$R = AB \sin i$$

$$r = AB \sin r$$



Dividing the former equation by the latter, gives:

$$\frac{\lambda'}{\lambda} = \frac{\sin i'}{\sin i}$$

Substituting equation 21 and dividing by  $\sin i$ , we obtain:

$$\lambda' = \lambda \left[ 1 - 2 \frac{v}{c} \sin \varphi \cos i + 2 \frac{v^2}{c^2} \sin^2 \varphi (2 \cos^2 i - 1) + \dots \right].$$

Since we shall have to divide the various distances by the corresponding wave lengths, it will be desirable to obtain the reciprocal of the above equation. Thus,

$$\frac{1}{\lambda'} = \frac{1}{\lambda} \left[ 1 + 2 \frac{v}{c} \sin \varphi \cos i + 2 \frac{v^2}{c^2} \sin^2 \varphi + \dots \right] \quad \text{Eqn. (23)}$$

Dividing the former equation by the latter, we have

$$\frac{\lambda}{\lambda} = \frac{\sin \frac{\lambda}{2}}{\sin \frac{\lambda}{2}}$$

Substituting equation (1) and dividing by  $\sin \frac{\lambda}{2}$ , we obtain

$$\lambda = \lambda \left[ 1 - \frac{\lambda^2}{2} \sin^2 \frac{\lambda}{2} + \frac{\lambda^4}{24} \sin^4 \frac{\lambda}{2} - \dots \right]$$

Since we shall have to divide the various coefficients by the same denominator, it will be convenient to obtain the reciprocal of the above series. Thus,

$$\frac{1}{\lambda} = \frac{1}{\lambda} \left[ 1 + \frac{\lambda^2}{2} \sin^2 \frac{\lambda}{2} + \dots \right] \quad \text{Eq. (2)}$$



### ANGLE OF REFLECTION OF THE RAY FROM PLATE M ALONG ARM S'

In figure I the apparatus is represented as moving in a general horizontal direction  $\underline{y}$ . Because of this motion, the ray of light leaving the source at O is not incident upon the plate M in a direction parallel to the source arm S.

If  $\alpha$  is the angle between the incident ray of light and S, we have by the sine law:

$$\frac{\sin \alpha}{vt} = \frac{\sin(\pi - \varphi)}{ct}.$$

from which we get:

$$\sin \alpha = \frac{v}{c} \sin \varphi \quad \text{Eqn. (24)}$$

and

$$\cos \alpha = (1 - \sin^2 \alpha)^{\frac{1}{2}}$$

$$\cos \alpha = 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \varphi + \dots \quad \text{Eqn. (25)}$$

If  $\beta$  is the angle between the normal to M and the source arm S, and  $i_1$  angle of incidence at the plate M then:

$$i_1 = \beta - \alpha.$$

Also

$$\beta = \frac{\pi}{2} - \varphi.$$

Therefore, substituting for  $\beta$ , we get:

$$i_1 = \frac{\pi}{2} - \varphi - \alpha \quad \text{Eqn. (26)}$$





To find the sine and the cosine of the angle of reflection, apply equations 21 and 22. Since the mirror is receding from the source of light the velocity is negative. The angle between the plate M and the direction of the velocity is  $(\theta + \phi)$ . Thus  $v \sin \phi$  must be replaced by  $-v \sin (\theta + \phi)$ . Hence, substituting in equations 21 and 22, we get:

$$\begin{aligned} \sin i_1' &= \sin i_1 + 2 \frac{v}{c} \sin(\theta + \phi) \sin i_1 \cos i_1 \\ &\quad + 2 \frac{v^2}{c^2} \sin^2(\theta + \phi) \sin i_1 (2 \cos^2 i_1 - 1) + \dots, \quad \text{Eqn. (27)} \end{aligned}$$

and

$$\begin{aligned} \cos i_1' &= \cos i_1 - 2 \frac{v}{c} \sin(\theta + \phi) \sin^2 i_1 \\ &\quad - 4 \frac{v^2}{c^2} \sin^2(\theta + \phi) \sin^2 i_1 \cos i_1 + \dots \quad \text{Eqn. (28)} \end{aligned}$$

Substituting equation 26 we get:

$$\begin{aligned} \sin i_1 &= \cos(\theta + \alpha) \\ &= \cos \theta \cos \alpha - \sin \theta \sin \alpha \end{aligned}$$

Substitute equations 24 and 25 .

$$\sin i_1 = \cos \theta - \frac{v}{c} \sin \theta \sin \phi - \frac{1}{2} \frac{v^2}{c^2} \cos \theta \sin^2 \phi + \dots$$

Again, substituting equation 26, we get:

$$\begin{aligned} \cos i_1 &= \sin(\theta + \alpha) \\ &= \sin \theta \cos \alpha + \cos \theta \sin \alpha \end{aligned}$$

Substitute equations 24, 25 .

$$\cos i_1 = \sin \theta + \frac{v}{c} \cos \theta \sin \phi - \frac{1}{2} \frac{v^2}{c^2} \sin \theta \sin^2 \phi + \dots$$

The following expressions will be developed only up to the power of  $\frac{v}{c}$  to which they will be needed when they are substituted in equations 27 and 28.

To find the sine and cosine of the angle of reflection, apply equations 11 and 12. Since the mirror is reflecting from the surface of the water, the angle between the plane of the mirror and the direction of the velocity is  $180^\circ$ . Thus  $\sin \theta = \sin 180^\circ = 0$  and  $\cos \theta = -1$ .

Then, substituting in equation 11 and 12, we get:

$$\sin i' = \sin i + \frac{v}{c} \sin \theta \cos \phi \cos \alpha$$

$$+ \frac{v}{c} \sin i \cos \theta \sin \phi \cos \alpha \quad (13)$$

and

$$\cos i' = \cos i - \frac{v}{c} \sin \theta \sin \phi \cos \alpha$$

$$- \frac{v}{c} \sin i \sin \theta \sin \phi \cos \alpha \quad (14)$$

Substituting equation 13 in 14, we get:

$$\sin i' = \cos i \cos \theta$$

$$= \cos \theta \cos i - \sin \theta \sin i$$

Substituting equation 13 in 15, we get:

$$\sin i' = \cos i - \frac{v}{c} \sin \theta \sin \phi \cos \alpha - \frac{v}{c} \sin i \cos \theta \sin \phi \cos \alpha$$

Substituting equation 13 in 16, we get:

$$\cos i' = \sin \theta \sin \phi \cos \alpha$$

$$= \sin \theta \cos \phi \sin \alpha$$

Substituting equation 13 in 17, we get:

$$\cos i' = \sin \theta \sin \phi \cos \alpha - \frac{v}{c} \sin \theta \sin \phi \cos \alpha$$

The following equations will be developed only up to the power

$\frac{v^2}{c^2}$ . They will be needed when they are substituted in equation 13 and



$$\sin i, \cos i, = \sin \theta \cos \theta + \frac{v}{c} \sin \phi (2 \cos^2 \theta - 1) + \dots$$

$$\sin^2 i, = \cos^2 \theta - 2 \frac{v}{c} \sin \phi \sin \theta \cos \theta + \dots$$

$$\sin i, \cos^2 i, = \sin^2 \theta \cos \theta + \dots$$

$$\sin^2 i, \cos i, = \sin \theta \cos^2 \theta + \dots$$

Substituting the appropriate expressions in equation 27, we obtain:

$$\begin{aligned} \sin i'_1 &= \cos \theta - \frac{v}{c} \sin \phi \sin \theta - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \phi \cos \theta \\ &+ 2 \frac{v}{c} \sin(\theta + \phi) [\sin \theta \cos \theta + \frac{v}{c} \sin \phi (2 \cos^2 \theta - 1)] \\ &+ 2 \frac{v^2}{c^2} \sin^2(\theta + \phi) [2 \sin^2 \theta \cos \theta - \cos \theta] + \dots \end{aligned}$$

Arranging according to powers of  $v/c$ , we get :

$$\begin{aligned} \sin i'_1 &= \cos \theta + \frac{v}{c} \sin \theta [2 \sin(\theta + \phi) \cos \theta - \sin \phi] \\ &+ \frac{v^2}{c^2} [2 \sin(\theta + \phi) \sin \phi (2 \cos^2 \theta - 1) - \frac{1}{2} \sin^2 \phi \cos \theta \\ &+ 2 \sin^2(\theta + \phi) \cos \theta (2 \sin^2 \theta - 1)] + \dots \quad \text{Eqn. (29)} \end{aligned}$$

Also, substituting the appropriate expressions in equation 28, we obtain:

$$\begin{aligned} \cos i'_1 &= \sin \theta + \frac{v}{c} \sin \phi \cos \theta - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \phi \sin \theta \\ &- 2 \frac{v}{c} \sin(\theta + \phi) [\cos^2 \theta - 2 \frac{v}{c} \sin \phi \sin \theta \cos \theta] \\ &- 4 \frac{v^2}{c^2} \sin^2(\theta + \phi) \sin \theta \cos^2 \theta + \dots \end{aligned}$$

Arranging according to powers of  $v/c$ , we get :

$$\begin{aligned} \cos i'_1 &= \sin \theta + \frac{v}{c} \cos \theta [\sin \phi - 2 \cos \theta \sin(\theta + \phi)] \\ &+ \frac{v^2}{c^2} [4 \sin(\theta + \phi) \sin \phi \sin \theta \cos \theta - \frac{1}{2} \sin^2 \phi \sin \theta \\ &- 4 \sin^2(\theta + \phi) \sin \theta \cos^2 \theta] + \dots \quad \text{Eqn. (30)} \end{aligned}$$

We shall find that the sine and the cosine of angle  $(i'_1 + \theta')$  will be needed, therefore we shall derive the equations for  $\sin(i'_1 + \theta')$  and  $\cos(i'_1 + \theta')$  at this point.

$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

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$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

Substituting the appropriate expressions in equation (2), we obtain

$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$[1 - \cos 2\theta] \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \sin^2 \theta$$

$$[1 - \cos 2\theta] \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \sin^2 \theta$$

According to equation (1), we get

$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$(2) \quad \sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

Also, substituting the appropriate expressions in equation (3), we

obtain

$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$[1 - \cos 2\theta] \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \sin^2 \theta$$

$$[1 - \cos 2\theta] \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \sin^2 \theta$$

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$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$\sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

$$(3) \quad \sin^2 \theta = \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

We shall find the value of  $\theta$  from equation (1) and

substituting the appropriate expressions in equation (2), we

obtain



$$\sin(i' + \theta') = \sin i' \cos \theta' + \cos i' \sin \theta'$$

Substitute equations 29 and 30.

$$\begin{aligned} \sin(i' + \theta') &= \cos \theta \cos \theta' + \frac{v}{c} \sin \theta \cos \theta' [2 \sin(\theta + \varphi) \cos \theta - \sin \varphi] \\ &\quad + \frac{v^2}{c^2} \cos \theta' [-\frac{1}{2} \sin^2 \varphi \cos \theta + 2 \sin(\theta + \varphi) \sin \varphi - 2 \sin^2(\theta + \varphi) \cos \theta \\ &\quad - 4 \sin(\theta + \varphi) \sin \varphi \sin^2 \theta + 4 \sin^2(\theta + \varphi) \sin^2 \theta \cos \theta] \\ &\quad + \sin \theta \sin \theta' - \frac{v}{c} \sin \theta' \cos \theta [2 \sin(\theta + \varphi) \cos \theta - \sin \varphi] \\ &\quad + \frac{v^2}{c^2} \sin \theta' [-\frac{1}{2} \sin^2 \varphi \sin \theta + 4 \sin(\theta + \varphi) \sin \theta \cos \theta \sin \varphi - 4 \sin^2(\theta + \varphi) \sin \theta \cos^2 \theta] \end{aligned}$$

Collecting terms and arranging according to powers of  $v/c$ , we get:

$$\begin{aligned} \sin(i' + \theta') &= \cos(\theta - \theta') + \frac{v}{c} \sin(\theta - \theta') [2 \sin(\theta + \varphi) \cos \theta - \sin \varphi] \\ &\quad + \frac{v^2}{c^2} [-\frac{1}{2} \sin^2 \varphi \cos(\theta - \theta') + 2 \sin(\theta + \varphi) \sin \varphi \cos \theta' \\ &\quad + \sin(\theta - \theta') \{4 \sin^2(\theta + \varphi) \sin \theta \cos \theta - 4 \sin(\theta + \varphi) \sin \varphi \sin \theta\} \\ &\quad - 2 \sin^2(\theta + \varphi) \cos \theta \cos \theta'] + \dots \end{aligned}$$

Substituting equations 4, 5, 9, 10, 11, 12, 13 and 14; we obtain:

$$\sin(i' + \theta') = 1 + \frac{v^2}{c^2} [-\frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\theta_0 + \varphi_0) \cos \theta_0 \{ \sin \varphi_0 - \sin(\theta_0 + \varphi_0) \cos \theta_0 \}] + \dots \quad \text{Eqn. (31)}$$

Similarly,

$$\cos(i' + \theta') = \cos i' \cos \theta' - \sin i' \sin \theta'$$

Substitute equations 29 and 30.

$$\begin{aligned} \cos(i' + \theta') &= \sin \theta \cos \theta' - \frac{v}{c} \cos \theta \cos \theta' [2 \sin(\theta + \varphi) \cos \theta - \sin \varphi] \\ &\quad + \frac{v^2}{c^2} \cos \theta' [-\frac{1}{2} \sin^2 \varphi \sin \theta + 4 \sin(\theta + \varphi) \sin \theta \cos \theta (\sin \varphi - \sin(\theta + \varphi) \cos \theta)] \\ &\quad - \cos \theta \sin \theta' - \frac{v}{c} \sin \theta \sin \theta' [2 \sin(\theta + \varphi) \cos \theta - \sin \varphi] \\ &\quad - \frac{v^2}{c^2} \sin \theta' [-\frac{1}{2} \sin^2 \varphi \cos \theta + 2 \sin(\theta + \varphi) \sin \varphi - 2 \sin^2(\theta + \varphi) \cos \theta \\ &\quad - 4 \sin(\theta + \varphi) \sin \varphi \sin^2 \theta + 4 \sin^2(\theta + \varphi) \sin^2 \theta \cos \theta] \end{aligned}$$

Collecting terms and arranging according to powers of  $v/c$ , we get

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

Substitute values of  $\theta$  and  $\phi$ .

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

Collecting terms and arranging according to powers of  $\sin\theta$  and  $\cos\theta$ .

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

Substituting values of  $\theta$  and  $\phi$ .

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

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$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

Collecting terms and arranging according to powers of  $\sin\theta$  and  $\cos\theta$ .



$$\begin{aligned}\cos(i_1' + \theta') &= \sin(\theta - \theta') - \frac{v}{c} (\cos(\theta - \theta')) [2 \sin(\theta + \varphi) \cos \theta - \sin \varphi] \\ &\quad + \frac{v^2}{c^2} \left[ -\frac{1}{2} \sin^2 \varphi \sin(\theta - \theta') - 2 \sin(\theta + \varphi) \sin \varphi \sin \theta' \right. \\ &\quad \left. + 4 \cos(\theta - \theta') \sin(\theta + \varphi) \sin \theta \{ \sin \varphi - \sin(\theta + \varphi) \cos \theta \} \right. \\ &\quad \left. + 2 \sin^2(\theta + \varphi) \cos \theta \sin \theta' \right] + \dots\end{aligned}$$

Substituting equations 4, 5, 9, 10, 11, 12, 13 and 14; we obtain:

$$\begin{aligned}\cos(i_1' + \theta') &= -\frac{v}{c} [2 \sin(\theta_0 + \varphi_0) \cos \theta_0 - \sin \varphi_0] \\ &\quad + \frac{v^2}{c^2} \left[ \sin \theta_0 \cos \theta_0 \{ \cos^2 \varphi_0 - \cos^2(\theta_0 + \varphi_0) \} \right. \\ &\quad \left. + \tan \theta_0 (\sin(\theta_0 + \varphi_0) \cos(\theta_0 + \varphi_0) - \sin \varphi_0 \cos \varphi_0) \right. \\ &\quad \left. + 2 \sin(\theta_0 + \varphi_0) \sin \theta_0 (\sin \varphi_0 - \sin(\theta_0 + \varphi_0) \cos \theta_0) \right] + \dots\end{aligned}$$

The second order term reduces to zero, hence, we get:

$$\begin{aligned}\cos(i_1' + \theta') &= -\frac{v}{c} [2 \sin(\theta_0 + \varphi_0) \cos \theta_0 - \sin \varphi_0] + \dots \\ &= -\frac{v}{c} \sin(2\theta_0 + \varphi_0) + \dots\end{aligned} \quad \text{Eqn. (32)}$$

#### ANGLE OF REFLECTION OF THE RAY FROM MIRROR M'

To find the sine and the cosine of the angle of reflection at M' apply equations 21 and 22. Since the mirror is approaching the source of light, the velocity is positive. The angle between the mirror M' and the direction of the velocity is  $(\theta + \theta' + \phi + \sigma - \frac{\pi}{2})$ , therefore,  $\sin \phi$  must be replaced by:

$$\sin(\theta + \theta' + \varphi + \sigma - \frac{\pi}{2}) = -\cos(\theta + \theta' + \varphi + \sigma).$$

Expanding the latter expression, we get:

$$\sin(\theta + \theta' + \varphi + \sigma - \frac{\pi}{2}) = -\cos(\theta + \theta' + \varphi) \cos \sigma + \sin(\theta + \theta' + \varphi) \sin \sigma.$$

Substituting equations 7, 8, 15, and 16 and observing that  $(v/c)^1$  is the





highest power to which this expression need be developed, we get :

$$\sin(\vartheta + \vartheta' + \sigma - \frac{\pi}{2}) = -\cos(2\vartheta_0 + \varphi_0) + \dots$$

The  $(v/c)^1$  term is equal to zero. Hence, substituting in equations 21 and 22, we get:

$$\begin{aligned} \sin i_1' &= \sin i_2 + 2\frac{v}{c} \cos(2\vartheta_0 + \varphi_0) \sin i_2 \cos i_2 \\ &+ 2\frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) \sin i_2 (2\cos^2 i_2 - 1) + \dots \end{aligned} \quad \text{Eqn. (33)}$$

and

$$\begin{aligned} \cos i_1' &= \cos i_2 - 2\frac{v}{c} \cos(2\vartheta_0 + \varphi_0) \sin^2 i_2 \\ &- 4\frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) \sin^2 i_2 \cos i_2 + \dots \end{aligned} \quad \text{Eqn. (34)}$$

To find  $i_2$  construct the lines AE and CK in figure I parallel to S'. Since they are bounded by the same parallel lines, they are both equal to S'. Then,

$$\psi = i_1' + \vartheta' - \frac{\pi}{2}.$$

Also,

$$\text{angle AEK} = \psi = i_2 - \sigma.$$

Substituting the latter expression for  $\psi$  in the former equation, we get :

$$i_2 = i_1' + \vartheta' + \sigma - \frac{\pi}{2}. \quad \text{Eqn. (35)}$$

Now

$$\sin i_2 = -\cos(i_1' + \vartheta' + \sigma) = -\cos(i_1' + \vartheta') \cos \sigma + \sin(i_1' + \vartheta') \sin \sigma$$

and

$$\cos i_2 = \sin(i_1' + \vartheta' + \sigma) = \sin(i_1' + \vartheta') \cos \sigma + \cos(i_1' + \vartheta') \sin \sigma.$$

Substitute equations 15, 16, This gives :

$$\sin i_2 = -\cos(i_1' + \vartheta') + \frac{v^2}{c^2} \cos(2\vartheta_0 + \varphi_0) \sin(2\vartheta_0 + \varphi_0) \sin(i_1' + \vartheta') + \dots$$

and

$$\cos i_2 = \sin(i_1' + \vartheta') + \frac{v^2}{c^2} \cos(2\vartheta_0 + \varphi_0) \sin(2\vartheta_0 + \varphi_0) \cos(i_1' + \vartheta') + \dots$$

$$\cos \delta' = \sin(\delta + \psi) + \frac{1}{2} \cos(2\delta + \psi) \sin(2\psi) \cos(\delta)$$

$$\sin \delta' = -\cos(\delta + \psi) + \frac{1}{2} \cos(2\delta + \psi) \sin(2\psi) \sin(\delta)$$

$$\cos \psi = \sin(\delta + \psi) \cos \delta + \cos(\delta + \psi) \sin \delta$$

$$\sin \psi = -\cos(\delta + \psi) \cos \delta + \sin(\delta + \psi) \sin \delta$$

$$\delta' = \delta + \psi + \frac{\pi}{2}$$

Substituting the latter expression for  $\delta$  in the former equation we get:

$$\sin \delta' = \sin \delta = \psi = \delta' - \delta$$

$$\psi = \delta' - \frac{\pi}{2}$$

It is thus seen that the angle  $\delta$  is not the angle  $\delta'$  but the angle  $\delta'$  minus  $\frac{\pi}{2}$ .

$$\cos \delta' = \cos \delta + \frac{1}{2} \cos(2\delta + \psi) \sin(2\psi)$$

$$\sin \delta' = \sin \delta + \frac{1}{2} \cos(2\delta + \psi) \sin(2\psi)$$

$$\sin \delta' = \sin \delta + \frac{1}{2} \cos(2\delta + \psi) \sin(2\psi)$$

$$\sin \delta' = \sin \delta + \frac{1}{2} \cos(2\delta + \psi) \sin(2\psi)$$

$$\sin \delta' = \sin \delta + \frac{1}{2} \cos(2\delta + \psi) \sin(2\psi)$$



Substitute equations 31 and 32 for  $\sin(i'_1 + \theta')$  and  $\cos(i'_1 + \theta')$ . This gives:

$$\sin i_2 = \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) + \frac{v^2}{c^2} \cos(2\vartheta_0 + \varphi_0) \sin(2\vartheta_0 + \varphi_0) + \dots, \quad \text{Eqn. (36)}$$

and

$$\cos i_2 = 1 + \frac{v^2}{c^2} \left[ -\frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \{ \sin \varphi_0 - \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \right] + \dots, \quad \text{Eqn. (37)}$$

The following expression will be developed only up to the power of  $v/c$  that it will be needed when it is substituted in equation 33.

$$\sin i_2 \cos i_2 = \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) + \dots, \quad \text{Eqn. (38)}$$

Since  $\sin i_2$  is of the order of magnitude of  $v/c$ , its appearance to the second power in the first order term of  $\cos i_2$  causes that term to be dropped; a fortiori in the second order term. Likewise, its appearance in the second order term of  $\sin i_2$  causes that term to be dropped.

Substituting equations 36 and 38 in equation 33 gives:

$$\begin{aligned} \sin i'_2 = & \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) + \frac{v^2}{c^2} \cos(2\vartheta_0 + \varphi_0) \sin(2\vartheta_0 + \varphi_0) \\ & + 2 \frac{v^2}{c^2} \cos(2\vartheta_0 + \varphi_0) \sin(2\vartheta_0 + \varphi_0) + \dots, \end{aligned}$$

which, on further simplification, becomes:

$$\sin i'_2 = \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) + 3 \frac{v^2}{c^2} \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) + \dots, \quad \text{Eqn. (39)}$$

Also, substituting equation 37 in equation 34, we get:

$$\begin{aligned} \cos i'_2 = & 1 + \frac{v^2}{c^2} \left[ -\frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \{ \sin \varphi_0 - \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \right] + \dots \\ \cos i'_2 = & 1 - \frac{v^2}{c^2} \left[ \frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \right] + \dots, \quad \text{Eqn. (40)} \end{aligned}$$





We shall find that the sine and the cosine of the angle  $(\theta' + i_2' + \sigma)$  will be needed, therefore we shall derive the equations for  $\sin(\theta' + i_2' + \sigma)$  and  $\cos(\theta' + i_2' + \sigma)$  at this point.

$$\sin(\theta' + i_2' + \sigma) = \sin(\theta' + i_2') \cos \sigma + \cos(\theta' + i_2') \sin \sigma$$

and

$$\cos(\theta' + i_2' + \sigma) = \cos(\theta' + i_2') \cos \sigma - \sin(\theta' + i_2') \sin \sigma$$

Substitute equations 15 and 16 for  $\sin \sigma$  and  $\cos \sigma$ .

$$\sin(\theta' + i_2' + \sigma) = \sin(\theta' + i_2') + \frac{v^2}{c^2} \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos(\theta' + i_2') + \dots \quad \text{Eqn. (41)}$$

and

$$\cos(\theta' + i_2' + \sigma) = \cos(\theta' + i_2') - \frac{v^2}{c^2} \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \sin(\theta' + i_2') + \dots \quad \text{Eqn. (42)}$$

Expand  $\sin(\theta' + i_2')$ .

$$\sin(\theta' + i_2') = \sin \vartheta' \cos i_2' + \cos \vartheta' \sin i_2'$$

Substitute equations 11, 12, 39 and 40.

$$\begin{aligned} \sin(\vartheta' + i_2') &= \sin \vartheta_0 \left[ 1 - \frac{v^2}{c^2} \{ \sin^2(2\vartheta_0 + \varphi_0) - \cos^2(\vartheta_0 + \varphi_0) \} \right] \\ &\quad \left[ 1 - \frac{v^2}{c^2} \left\{ \frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \right\} \right] \\ &\quad + \cos \vartheta_0 \left[ \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) + 3 \frac{v^2}{c^2} \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \right] + \dots \end{aligned}$$

Multiply and arrange according to powers of  $v/c$ .

$$\begin{aligned} \sin(\vartheta' + i_2') &= \sin \vartheta_0 + \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \\ &\quad + \frac{v^2}{c^2} \left[ -\frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 - \frac{1}{2} \sin^2 \varphi_0 \sin \vartheta_0 \right. \\ &\quad \left. - 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 \right. \\ &\quad \left. + 3 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \right] \end{aligned}$$

We shall find that the sine and the cosine of the angle  $(\theta' + \frac{1}{2}\pi)$  will be needed, therefore we shall derive the equations for  $\sin(\theta' + \frac{1}{2}\pi)$  and  $\cos(\theta' + \frac{1}{2}\pi)$  at this point.

$$\sin(\theta' + \frac{1}{2}\pi) = \sin(\theta' + \frac{1}{2}\pi) \cos \theta + \cos(\theta' + \frac{1}{2}\pi) \sin \theta$$

and

$$\cos(\theta' + \frac{1}{2}\pi) = \cos(\theta' + \frac{1}{2}\pi) \cos \theta - \sin(\theta' + \frac{1}{2}\pi) \sin \theta$$

Substitute equations 13 and 14 for  $\sin \theta$  and  $\cos \theta$ .

$$\sin(\theta' + \frac{1}{2}\pi) = \sin(\theta' + \frac{1}{2}\pi) \left[ \cos \theta + \frac{\sqrt{2}}{2} \sin \theta \right] + \cos(\theta' + \frac{1}{2}\pi) \left[ \sin \theta + \frac{\sqrt{2}}{2} \cos \theta \right] \quad \text{Eqn. (15)}$$

and

$$\cos(\theta' + \frac{1}{2}\pi) = \cos(\theta' + \frac{1}{2}\pi) \left[ \cos \theta + \frac{\sqrt{2}}{2} \sin \theta \right] - \sin(\theta' + \frac{1}{2}\pi) \left[ \sin \theta + \frac{\sqrt{2}}{2} \cos \theta \right] \quad \text{Eqn. (16)}$$

Expand  $\sin(\theta' + \frac{1}{2}\pi)$ .

$$\sin(\theta' + \frac{1}{2}\pi) = \sin \theta' \cos \frac{1}{2}\pi + \cos \theta' \sin \frac{1}{2}\pi$$

Substitute equations 11, 12, 32 and 40.

$$\sin(\theta' + \frac{1}{2}\pi) = \sin \theta' \left[ \frac{1}{2} - \frac{\sqrt{2}}{2} \right] + \cos \theta' \left[ \frac{1}{2} + \frac{\sqrt{2}}{2} \right]$$

$$\left[ \frac{1}{2} - \frac{\sqrt{2}}{2} \right] \sin \theta' + \left[ \frac{1}{2} + \frac{\sqrt{2}}{2} \right] \cos \theta' = \sin(\theta' + \frac{1}{2}\pi) \cos \theta + \cos(\theta' + \frac{1}{2}\pi) \sin \theta$$

Similarly and arrange according to powers of  $\sin \theta$ .

$$\sin(\theta' + \frac{1}{2}\pi) = \sin \theta' \left[ \frac{1}{2} + \frac{\sqrt{2}}{2} \right] + \cos \theta' \left[ \frac{1}{2} - \frac{\sqrt{2}}{2} \right]$$

$$\left[ \frac{1}{2} + \frac{\sqrt{2}}{2} \right] \sin \theta' + \left[ \frac{1}{2} - \frac{\sqrt{2}}{2} \right] \cos \theta' = \sin(\theta' + \frac{1}{2}\pi) \cos \theta + \cos(\theta' + \frac{1}{2}\pi) \sin \theta$$

$$- \sin \theta' \cos \theta + \cos \theta' \sin \theta = \sin(\theta' + \frac{1}{2}\pi) \cos \theta + \cos(\theta' + \frac{1}{2}\pi) \sin \theta$$

$$[ \sin \theta' \cos \theta + \cos \theta' \sin \theta ] = [ \sin(\theta' + \frac{1}{2}\pi) \cos \theta + \cos(\theta' + \frac{1}{2}\pi) \sin \theta ]$$



Similarly,

$$\cos(\theta' + i_2') = \cos \theta' \cos i_2' - \sin \theta' \sin i_2'$$

Again, substituting equations 11, 12, 39 and 40; we get:

$$\begin{aligned} \cos(\theta' + i_2') &= \cos \theta_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2(2\theta_0 + \phi_0) + \cos(\theta_0 + \phi_0) [2 \tan \theta_0 \sin(\theta_0 + \phi_0) - \cos(\theta_0 + \phi_0)] \} \right] \\ &\quad \left[ 1 - \frac{v^2}{c^2} \left\{ \frac{1}{2} \sin^2 \phi_0 + 2 \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin \theta_0 \cos \theta_0 \right\} \right] \\ &\quad - \sin \theta_0 \left[ \frac{v}{c} \sin(2\theta_0 + \phi_0) + 3 \frac{v^2}{c^2} \sin(2\theta_0 + \phi_0) \cos(2\theta_0 + \phi_0) \right] + \dots \end{aligned}$$

Multiply and arrange according to powers of  $v/c$ .

$$\begin{aligned} \cos(\theta' + i_2') &= \cos \theta_0 - \frac{v}{c} \sin(2\theta_0 + \phi_0) \sin \theta_0 \\ &\quad + \frac{v^2}{c^2} \left[ \frac{1}{2} \cos^2(2\theta_0 + \phi_0) \cos \theta_0 + \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin \theta_0 \right. \\ &\quad \left. - \frac{1}{2} \cos^2(\theta_0 + \phi_0) \cos \theta_0 - \frac{1}{2} \sin^2 \phi_0 \cos \theta_0 \right. \\ &\quad \left. - 2 \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin \theta_0 \cos^2 \theta_0 \right. \\ &\quad \left. - 3 \sin(2\theta_0 + \phi_0) \cos(2\theta_0 + \phi_0) \sin \theta_0 \right] + \dots \end{aligned}$$

Substituting for  $\sin(\theta' + i_2')$  and  $\cos(\theta' + i_2')$  in equation 41, we get:

$$\begin{aligned} \sin(\theta' + i_2' + \sigma) &= \sin \theta_0 + \frac{v}{c} \sin(2\theta_0 + \phi_0) \cos \theta_0 \\ &\quad + \frac{v^2}{c^2} \left[ -\frac{1}{2} \sin^2(2\theta_0 + \phi_0) \sin \theta_0 + \frac{1}{2} \cos^2(\theta_0 + \phi_0) \sin \theta_0 \right. \\ &\quad \left. - \frac{1}{2} \sin^2 \phi_0 \sin \theta_0 \right. \\ &\quad \left. - 2 \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin^2 \theta_0 \cos \theta_0 \right. \\ &\quad \left. + 4 \sin(2\theta_0 + \phi_0) \cos(2\theta_0 + \phi_0) \cos \theta_0 \right] + \dots \quad \text{Eqn. (43)} \end{aligned}$$

For later use, it will be convenient to write this equation symbolically as:

$$\sin(\theta' + i_2' + \sigma) = E + \frac{v}{c} F + \frac{v^2}{c^2} P + \dots \quad \text{Eqn. (44)}$$

where E, F and P are defined as the coefficients of the  $(v/c)^0$ ,  $(v/c)^1$  and  $(v/c)^2$  terms respectively.

Equation (10)

$$\cos(\theta + i) = \cos \theta \cos i - \sin \theta \sin i$$

Again, substituting equations 11, 12, 13 and 14, we get

$$\begin{aligned} \cos(\theta + i) &= \cos \theta \left[ 1 + \frac{1}{2} \left( \frac{v}{c} \right)^2 \cos^2 \theta + \cos \theta \left( \frac{v}{c} \right) \sin \theta \right] - \sin \theta \left[ \frac{v}{c} \sin \theta + \frac{1}{2} \left( \frac{v}{c} \right)^2 \sin^2 \theta + \dots \right] \\ &= \cos \theta \left[ 1 + \frac{1}{2} \left( \frac{v}{c} \right)^2 \cos^2 \theta + \cos \theta \left( \frac{v}{c} \right) \sin \theta \right] - \sin \theta \left[ \frac{v}{c} \sin \theta + \frac{1}{2} \left( \frac{v}{c} \right)^2 \sin^2 \theta + \dots \right] \end{aligned}$$

Similarly and arrange according to powers of  $v/c$ .

$$\begin{aligned} \cos(\theta + i) &= \cos \theta - \frac{v}{c} \sin \theta \sin \theta + \frac{1}{2} \left( \frac{v}{c} \right)^2 \cos^2 \theta - \frac{1}{2} \left( \frac{v}{c} \right)^2 \sin^2 \theta + \dots \\ &= \cos \theta - \frac{v}{c} \sin^2 \theta + \frac{1}{2} \left( \frac{v}{c} \right)^2 (\cos^2 \theta - \sin^2 \theta) + \dots \end{aligned}$$

Substituting for  $\sin^2 \theta$  and  $\cos^2 \theta$  in equation 11, we get

$$\begin{aligned} \cos(\theta + i) &= \cos \theta - \frac{v}{c} \sin^2 \theta + \frac{1}{2} \left( \frac{v}{c} \right)^2 (\cos^2 \theta - \sin^2 \theta) + \dots \\ &= \cos \theta - \frac{v}{c} \sin^2 \theta + \frac{1}{2} \left( \frac{v}{c} \right)^2 (\cos^2 \theta - \sin^2 \theta) + \dots \end{aligned}$$

For later use, it will be convenient to write this equation symbolically as

$$\cos(\theta + i) = \cos \theta - \frac{v}{c} \sin^2 \theta + \frac{1}{2} \left( \frac{v}{c} \right)^2 (\cos^2 \theta - \sin^2 \theta) + \dots \quad (15)$$

where  $\cos \theta$  and  $\sin \theta$  are defined as the coefficients of the  $(v/c)^0$  and

$(v/c)^2$  terms respectively.



Similarly, making the proper substitutions in equation 42, we get:

$$\begin{aligned}\cos(\theta' + i_2' + \sigma) &= \cos \theta_0 - \frac{v}{c} \sin(2\theta_0 + \phi_0) \sin \theta_0 \\ &+ \frac{v^2}{c^2} \left[ \frac{1}{2} \cos^2(2\theta_0 + \phi_0) \cos \theta_0 + \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin \theta_0 \right. \\ &- \frac{1}{2} \cos^2(\theta_0 + \phi_0) \cos \theta_0 - \frac{1}{2} \sin^2 \phi_0 \cos \theta_0 \\ &- 2 \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin \theta_0 \cos^2 \theta_0 \\ &\left. - 4 \sin(2\theta_0 + \phi_0) \cos(2\theta_0 + \phi_0) \sin \theta_0 \right] + \dots \quad \text{Eqn. (45)}\end{aligned}$$

Also, we shall find that the reciprocal of  $\sin(\theta' + i_2' + \sigma)$  will be needed up to the first power of  $v/c$ .

$$\frac{1}{\sin(\theta' + i_2' + \sigma)} = \frac{1}{\sin \theta_0} - \frac{v}{c} \frac{\cos \theta_0}{\sin^2 \theta_0} \sin(2\theta_0 + \phi_0) + \dots \quad \text{Eqn. (46)}$$

#### ANGLE OF REFLECTION OF THE RAY FROM THE MIRROR M''

To find the sine and the cosine of the angle of reflection at mirror M'' apply equations 21 and 22. Since the mirror is receding from the source of light, the velocity is negative. The angle between M'' and the direction of the velocity is  $(\frac{\pi}{2} - \phi + \omega)$ , therefore,  $\sin \phi$  must be replaced by  $\sin(\frac{\pi}{2} - \phi + \omega)$  or  $\cos(\phi - \omega)$ . Hence:

$$\begin{aligned}\sin j_1' &= \sin j_1 + 2 \frac{v}{c} \cos(\phi - \omega) \sin j_1 \cos j_1 \\ &+ 2 \frac{v^2}{c^2} \cos^2(\phi - \omega) \sin j_1 (2 \cos^2 j_1 - 1) + \dots, \quad \text{Eqn. (47)}\end{aligned}$$

and

$$\begin{aligned}\cos j_1' &= \cos j_1 - 2 \frac{v}{c} \cos(\phi - \omega) \sin^2 j_1 \\ &- 4 \frac{v^2}{c^2} \cos^2(\phi - \omega) \sin^2 j_1 \cos j_1 + \dots, \quad \text{Eqn. (48)}\end{aligned}$$

Similarly, making the proper substitutions in equation 42, we get:

$$\begin{aligned} \cos(\theta + \theta_0) &= \cos \theta_0 - \frac{v}{c} \sin(\theta_0 + \phi_0) \sin \theta_0 \\ &+ \frac{v}{c} \left[ \frac{1}{2} \cos^2(\theta_0 + \phi_0) \cos \theta_0 + \sin(\theta_0 + \phi_0) \cos \theta_0 \sin \theta_0 \right. \\ &\left. - \frac{1}{2} \cos^2(\theta_0 + \phi_0) \cos \theta_0 - \frac{1}{2} \sin^2(\theta_0 + \phi_0) \cos \theta_0 \right] \\ &- \frac{1}{2} \sin^2(\theta_0 + \phi_0) \cos \theta_0 \sin \theta_0 \cos \theta_0 \\ &- \frac{1}{2} \sin^2(\theta_0 + \phi_0) \cos \theta_0 \sin \theta_0 \cos \theta_0 \end{aligned} \quad \text{Eqn. (43)}$$

Also, we shall find that the reciprocal of  $\sin(\theta + \theta_0)$  will be needed up to the first power of  $v/c$ .

$$\frac{1}{\sin(\theta + \theta_0)} = \frac{1}{\sin \theta_0} - \frac{v \cos \theta_0}{c \sin^2 \theta_0} \sin(\theta_0 + \phi_0) + \dots \quad \text{Eqn. (44)}$$

### TABLE OF REFLECTION OF THE RAY FROM THE MIRROR

To find the change in the angle of reflection of mirror M, apply equations 41 and 42. Since the mirror is receding from the source of light, the velocity is negative. The angle between M and the direction of the velocity is  $(\frac{\pi}{2} - \theta_0)$ , therefore,  $\sin \theta_0$  must be replaced by  $\sin(\frac{\pi}{2} - \theta_0)$  or  $\cos(\theta_0)$ . Hence:

$$\begin{aligned} \sin \theta_0 &= \sin \theta_0 + \frac{v}{c} \cos(\theta_0 + \phi_0) \sin \theta_0 \\ &+ \frac{1}{2} \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0) \sin \theta_0 \cos \theta_0 + \dots \end{aligned} \quad \text{Eqn. (45)}$$

and

$$\begin{aligned} \cos \theta_0 &= \cos \theta_0 - \frac{v}{c} \cos(\theta_0 + \phi_0) \sin \theta_0 \\ &- \frac{1}{2} \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0) \sin \theta_0 \cos \theta_0 + \dots \end{aligned} \quad \text{Eqn. (46)}$$



Now

$$\cos(\varphi - \omega) = \cos \varphi \cos \omega + \sin \varphi \sin \omega$$

Substituting equations 3, 4, 17 and 18 and observing that  $\cos(\varphi - \omega)$  need be developed only up to the first power of  $v/c$ , we obtain:

$$\cos(\varphi - \omega) = \cos \varphi_0 + \frac{v}{c} \sin \varphi_0 \sin \mu_0 + \dots$$

Also:

$$\cos^2(\varphi - \omega) = \cos^2 \varphi_0 + \dots$$

to the order of approximation to which it is needed.

Referring to figure I we see that:

$$j_1 = \alpha - \omega$$

Therefore,

$$\begin{aligned} \sin j_1 &= \sin(\alpha - \omega) \\ &= \sin \alpha \cos \omega - \cos \alpha \sin \omega \end{aligned}$$

Substituting equations 17, 18, 24 and 25 gives:

$$\sin j_1 = \frac{v}{c} (\sin \varphi_0 - \sin \mu_0) - \frac{v^2}{c^2} \sin \varphi_0 \cos \varphi_0 + \dots$$

Similarly:

$$\begin{aligned} \cos j_1 &= \cos(\alpha - \omega) \\ &= \cos \alpha \cos \omega + \sin \alpha \sin \omega \end{aligned}$$

Making the same substitutions as in the previous case, we get:

$$\cos j_1 = 1 - \frac{1}{2} \frac{v^2}{c^2} [\sin^2 \varphi_0 + \sin^2 \mu_0 - 2 \sin \varphi_0 \sin \mu_0] + \dots$$

The following expression will be developed only up to the power of

$$\cos(\phi - \omega) = \cos \phi \cos \omega + \sin \phi \sin \omega$$

Substituting equations 3, 4, 19 and 20 and observing that  $\cos(\phi - \omega)$  can be developed only up to the first power of  $\omega$ , we obtain

$$\cos(\phi - \omega) = \cos \phi + \frac{\omega}{2} \sin 2\phi$$

$$\cos(\phi - \omega) = \cos \phi + \frac{\omega}{2} \sin 2\phi$$

Substituting this expression for  $\cos(\phi - \omega)$  in equation 2

gives us the following result:

$$j = \omega + \frac{\omega^2}{2} \sin 2\phi$$

$$\sin j = \sin(\omega + \frac{\omega^2}{2} \sin 2\phi)$$

$$= \sin \omega \cos \frac{\omega^2}{2} \sin 2\phi + \cos \omega \sin \frac{\omega^2}{2} \sin 2\phi$$

Substituting equations 17, 18, 19 and 20 gives

$$\sin j = \frac{\omega}{2} \sin 2\phi + \frac{\omega^2}{2} \sin 2\phi \cos 2\phi$$

$$\cos j = \cos(\omega + \frac{\omega^2}{2} \sin 2\phi)$$

$$= \cos \omega \cos \frac{\omega^2}{2} \sin 2\phi + \sin \omega \sin \frac{\omega^2}{2} \sin 2\phi$$

Using the same method as in the previous case, we get

$$\cos j = 1 - \frac{\omega^2}{2} \sin^2 2\phi + \frac{\omega^3}{2} \sin 2\phi \cos 2\phi$$

The following expression will be developed only up to the first power



$v/c$  to which it will be needed when it is substituted in equation 41.

$$\sin j, \cos j, = \frac{v}{c} (\sin \varphi_0 - \sin \mu_0) + \dots$$

The remark made concerning  $\sin i_2$ , on page 43, may likewise be made about  $\sin j_1$ . Hence, making the proper substitutions in equations 47 and 48, we get:

$$\sin j'_1 = \frac{v}{c} (\sin \varphi_0 - \sin \mu_0) + \frac{v^2}{c^2} \cos \varphi_0 (\sin \varphi_0 - 2 \sin \mu_0) + \dots \quad \text{Eqn. (49)}$$

and

$$\cos j'_1 = 1 - \frac{1}{2} \frac{v^2}{c^2} (\sin^2 \varphi_0 + \sin^2 \mu_0 - 2 \sin \varphi_0 \sin \mu_0) + \dots \quad \text{Eqn. (50)}$$

We shall find that the sine and the cosine of the angle  $(\theta - j'_1)$  will be needed, therefore, we shall derive the equations for  $\sin(\theta - j'_1)$  and  $\cos(\theta - j'_1)$  at this point.

$$\sin(\theta - j'_1) = \sin \theta \cos j'_1 - \cos \theta \sin j'_1$$

and

$$\cos(\theta - j'_1) = \cos \theta \cos j'_1 + \sin \theta \sin j'_1$$

Substituting equations 9, 10, 49 and 50 in the former expression, gives:

$$\begin{aligned} \sin(\theta - j'_1) = & \sin \theta_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \sin^2(\theta_0 + \varphi_0) - \cos^2 \varphi_0 \} \right. \\ & \left. \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \sin^2 \varphi_0 + \sin^2 \mu_0 - 2 \sin \varphi_0 \sin \mu_0 \} \right] \right. \\ & \left. - \cos \theta_0 \left[ \frac{v}{c} \{ \sin \varphi_0 - \sin \mu_0 \} + \frac{v^2}{c^2} \cos \varphi_0 \{ \sin \varphi_0 - 2 \sin \mu_0 \} \right] + \dots \right] \end{aligned}$$

Multiply and arrange according to powers of  $v/c$ .

$$\begin{aligned} \sin(\theta - j'_1) = & \sin \theta_0 - \frac{v}{c} \cos \theta_0 [\sin \varphi_0 - \sin \mu_0] \\ & + \frac{v^2}{c^2} \left[ \frac{1}{2} \sin \theta_0 \{ \sin^2(\theta_0 + \varphi_0) - \cos^2 \varphi_0 + \sin^2 \varphi_0 + \sin^2 \mu_0 - 2 \sin \varphi_0 \sin \mu_0 \} \right. \\ & \left. + \cos \theta_0 \cos \varphi_0 \{ \sin \varphi_0 - 2 \sin \mu_0 \} \right] + \dots \quad \text{Eqn. (51)} \end{aligned}$$

to which it will be referred when it is substituted in equation 41.

$$2\sin(\theta/2) = \frac{1}{2}(\sin\theta + \sin\theta) + \dots$$

The result is also consistent with the result on page 43, and likewise the result on page 44. Hence, making the proper substitutions in equations 47 and 48, we get:

$$\sin(\theta/2) = \frac{1}{2}(\sin\theta + \sin\theta) + \frac{1}{2}(\sin\theta + \sin\theta) + \dots \quad \text{Eq. (49)}$$

$$\cos(\theta/2) = 1 - \frac{1}{2}(\cos\theta + \cos\theta) + \frac{1}{2}(\cos\theta + \cos\theta) + \dots \quad \text{Eq. (50)}$$

We shall find that the value of the cosine of the angle  $(\theta/2)$  will be equal to the value of the cosine of the angle  $(\theta/2)$  and the value of the cosine of the angle  $(\theta/2)$  will be equal to the value of the cosine of the angle  $(\theta/2)$ .

$$2\sin(\theta/2) = \sin\theta + \sin\theta + \dots$$

$$\cos(\theta/2) = \cos\theta + \cos\theta + \dots$$

Substituting equations 49, 50, and 51 in the former expression, gives:

$$\sin(\theta/2) = \frac{1}{2}(\sin\theta + \sin\theta) + \frac{1}{2}(\sin\theta + \sin\theta) + \dots$$

$$1 - \frac{1}{2}(\cos\theta + \cos\theta) + \frac{1}{2}(\cos\theta + \cos\theta) + \dots$$

$$- \cos(\theta/2) = \frac{1}{2}(\cos\theta + \cos\theta) + \frac{1}{2}(\cos\theta + \cos\theta) + \dots$$

Adding and subtracting to give:

$$2\sin(\theta/2) = \sin\theta + \sin\theta + \dots$$

$$- \cos(\theta/2) = \cos\theta + \cos\theta + \dots$$

$$+ \cos(\theta/2) = \cos\theta + \cos\theta + \dots$$

Eq. (51)



Again, making the same substitutions in the expression for  $\cos(\theta-j'_1)$ , we get:

$$\begin{aligned}\cos(\theta-j'_1) = & \cos \vartheta_0 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \{ \cos^2(\vartheta_0 + \varphi_0) + \cos \varphi_0 (2 \tan \vartheta_0 \sin \varphi_0 - \cos \varphi_0) \} \right] \\ & \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \sin^2 \varphi_0 + \sin^2 \mu_0 - 2 \sin \varphi_0 \sin \mu_0 \} \right] \\ & + \sin \vartheta_0 \left[ \frac{v}{c} (\sin \varphi_0 - \sin \mu_0) + \frac{v^2}{c^2} \cos \varphi_0 (\sin \varphi_0 - 2 \sin \mu_0) \right] + \dots\end{aligned}$$

Multiply and arrange according to powers  $v/c$ .

$$\begin{aligned}\cos(\theta-j'_1) = & \cos \vartheta_0 + \frac{v}{c} \sin \vartheta_0 (\sin \varphi_0 - \sin \mu_0) \\ & + \frac{v^2}{c^2} \left[ \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) + 2 \sin \vartheta_0 \sin \varphi_0 \cos \varphi_0 - \frac{1}{2} \cos \vartheta_0 \right. \\ & \left. + \cos \vartheta_0 \sin \varphi_0 \sin \mu_0 - 2 \sin \vartheta_0 \cos \varphi_0 \sin \mu_0 \right. \\ & \left. - \frac{1}{2} \cos \vartheta_0 \sin^2 \mu_0 \right] + \dots, \quad \text{Eqn. (52)}\end{aligned}$$

#### ANGLE OF REFLECTION FROM PLATE M AFTER RETURN FROM MIRROR M''

To find the sine and the cosine of the angle of reflection at the plate M after the ray has been reflected from M'' apply equations 21 and 22. Since the mirror is approaching the source of light, the velocity is positive. The angle between the plate M and the direction of the velocity is  $(\theta + \phi)$ , therefore, substituting  $(\theta + \phi)$  for  $\phi$ , we get:

$$\begin{aligned}\sin j'_2 = & \sin j_2 - 2 \frac{v}{c} \sin(\vartheta + \varphi) \sin j_2 \cos j_2 \\ & + 2 \frac{v^2}{c^2} \sin^2(\vartheta + \varphi) \sin j_2 (2 \cos^2 j_2 - 1) + \dots, \quad \text{Eqn. (53)}\end{aligned}$$

and

$$\begin{aligned}\cos j'_2 = & \cos j_2 + 2 \frac{v}{c} \sin(\vartheta + \varphi) \sin^2 j_2 \\ & - 4 \frac{v^2}{c^2} \sin^2(\vartheta + \varphi) \sin^2 j_2 \cos j_2 + \dots\end{aligned}$$

Eqn. (54)

again, making the same substitutions in the expression for  $\cos(\theta - \frac{1}{2})$ , we get:

$$\cos(\theta - \frac{1}{2}) = \cos \theta_0 \left[ 1 + \frac{v}{c} \left\{ \cos^2(\theta_0 + \varphi_0) + \cos \varphi_0 (2 \sin \theta_0 \sin \varphi_0 - \cos \theta_0) \right\} \right. \\ \left. \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \sin^2 \varphi_0 + 2 \sin \theta_0 \sin \varphi_0 - 2 \sin^2 \theta_0 \} \right] \right. \\ \left. + \sin \theta_0 \left[ \frac{v}{c} (2 \sin \theta_0 \sin \varphi_0 - 2 \sin^2 \theta_0) + \frac{v^2}{c^2} \cos \varphi_0 (2 \sin \theta_0 \sin \varphi_0 - 2 \sin^2 \theta_0) \right] \right] + \dots$$

Multiply and arrange according to powers  $v/c$ .

$$\cos(\theta - \frac{1}{2}) = \cos \theta_0 + \frac{v}{c} \sin \theta_0 (2 \sin \theta_0 \sin \varphi_0 - 2 \sin^2 \theta_0) \\ + \frac{v^2}{c^2} \left[ \frac{1}{2} \cos^2(\theta_0 + \varphi_0) + 2 \sin \theta_0 \sin \varphi_0 \cos \theta_0 - \frac{1}{2} \cos^2 \theta_0 \right. \\ \left. + \cos \theta_0 \sin \varphi_0 \sin \theta_0 - 2 \sin \theta_0 \sin \varphi_0 \cos \theta_0 \right. \\ \left. - \frac{1}{2} \cos^2 \theta_0 \sin^2 \varphi_0 + \dots \right] \quad \text{Eqn. (22)}$$

ANGLE OF REFLECTION FROM PLATE M AFTER REFLECTION FROM MIRROR M'

To find the sine and the cosine of the angle of reflection of the plate M after the ray has been reflected from M' apply equations 21 and 22. Since the mirror is approaching the source of light, the velocity is positive. The angle between the plate M and the direction of the velocity is  $(\theta + \frac{1}{2})$ . Therefore, substituting  $(\theta + \frac{1}{2})$  for  $\theta$  we get:

$$\sin \theta_1 = \sin \theta_2 - \frac{v}{c} \sin(\theta_2 + \varphi_2) \sin \theta_2 \cos \theta_2 \\ + \frac{v^2}{c^2} \sin^2(\theta_2 + \varphi_2) \sin \theta_2 (2 \cos \theta_2 - 1) + \dots \quad \text{Eqn. (23)}$$

$$\cos \theta_1 = \cos \theta_2 + \frac{v}{c} \sin(\theta_2 + \varphi_2) \sin \theta_2 \\ - \frac{v^2}{c^2} \sin^2(\theta_2 + \varphi_2) \sin \theta_2 \cos \theta_2 + \dots \quad \text{Eqn. (24)}$$



Referring to figure I, we observe that:

$$j_2 = \frac{\pi}{2} - \vartheta - \omega + j_1'$$

Therefore,

$$\begin{aligned}\sin j_2 &= \cos(\vartheta + \omega - j_1') \\ &= \cos(\vartheta - j_1') \cos \omega - \sin(\vartheta - j_1') \sin \omega\end{aligned}$$

and

$$\begin{aligned}\cos j_2 &= \sin(\vartheta + \omega - j_1') \\ &= \sin(\vartheta - j_1') \cos \omega + \cos(\vartheta - j_1') \sin \omega\end{aligned}$$

Substituting equations 17 and 18 in expression for  $\sin j_2$ , we get:

$$\begin{aligned}\sin j_2 &= \cos(\vartheta - j_1') - \frac{\nu}{c} \sin \mu_0 \sin(\vartheta - j_1') \\ &\quad - \frac{\nu^2}{c^2} \left[ \frac{1}{2} \sin^2 \mu_0 \cos(\vartheta - j_1') + \sin \varphi_0 \cos \varphi_0 \sin(\vartheta - j_1') \right] + \dots\end{aligned}$$

Now, replacing  $\sin(\vartheta - j_1')$  and  $\cos(\vartheta - j_1')$  by equations 51 and 52, respectively, we get:

$$\begin{aligned}\sin j_2 &= \cos \vartheta_0 + \frac{\nu}{c} \sin \vartheta_0 [\sin \varphi_0 - 2 \sin \mu_0] \\ &\quad + \frac{\nu^2}{c^2} \left[ \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \sin \vartheta_0 \sin \varphi_0 \cos \varphi_0 \right. \\ &\quad \left. - \frac{1}{2} \cos \vartheta_0 + 2 \cos \vartheta_0 \sin \varphi_0 \sin \mu_0 \right. \\ &\quad \left. - 2 \sin \vartheta_0 \cos \varphi_0 \sin \mu_0 - 2 \cos \vartheta_0 \sin^2 \mu_0 \right] + \dots\end{aligned}$$

Likewise, substituting equations 17 and 18 in expression for  $\cos j_2$ , we get:

$$\begin{aligned}\cos j_2 &= \sin(\vartheta - j_1') + \frac{\nu}{c} \sin \mu_0 \cos(\vartheta - j_1') \\ &\quad - \frac{\nu^2}{c^2} \left[ \frac{1}{2} \sin^2 \mu_0 \sin(\vartheta - j_1') - \sin \varphi_0 \cos \varphi_0 \cos(\vartheta - j_1') \right] + \dots\end{aligned}$$

Again, replacing  $\sin(\vartheta - j_1')$  and  $\cos(\vartheta - j_1')$  by equations 51 and 52, respectively,

Referring to figure 1, we observe that:

$$j_2 = \omega - \theta - \frac{\pi}{2} = j_1$$

Therefore,

$$\sin j_2 = \cos(j_1 + \frac{\pi}{2})$$

$$= \cos(j_1) \cos \frac{\pi}{2} - \sin(j_1) \sin \frac{\pi}{2}$$

and

$$\cos j_1 = \sin(j_1 + \frac{\pi}{2})$$

$$= \sin(j_1) \cos \frac{\pi}{2} + \cos(j_1) \sin \frac{\pi}{2}$$

Substituting equations 17 and 18 in expression for  $j_2$ , we get:

$$\sin j_2 = \cos(j_1) \cos \frac{\pi}{2} - \frac{\pi}{2} \sin j_1 \sin \frac{\pi}{2}$$

$$= -\frac{\pi}{2} \left[ \frac{1}{2} \sin^2 \theta \cos \phi \cos \frac{\pi}{2} + \sin \theta \cos \phi \sin \frac{\pi}{2} \right] + \dots$$

Now, replacing  $\sin(\theta - \frac{\pi}{2})$  and  $\cos(\theta - \frac{\pi}{2})$  by equations 21 and 22, respectively,

we get:

$$\sin j_2 = \cos \theta + \frac{\pi}{2} \sin \theta \cos \phi \sin \frac{\pi}{2}$$

$$+ \frac{\pi^2}{8} \left[ \frac{1}{2} \cos^2 \theta \cos \phi \cos \frac{\pi}{2} + \sin \theta \cos \phi \sin \frac{\pi}{2} \right]$$

$$- \frac{1}{2} \cos^2 \theta + 2 \cos \theta \sin \theta \sin \phi \sin \frac{\pi}{2}$$

$$- 2 \sin \theta \cos \phi \sin \theta \sin \frac{\pi}{2} - 2 \cos \theta \sin \theta \sin \phi \sin \frac{\pi}{2} + \dots$$

Similarly, substituting equations 17 and 18 in expression for  $\cos j_2$ ,

we get:

$$\cos j_2 = \sin(j_1) \cos \frac{\pi}{2} + \frac{\pi}{2} \sin j_1 \sin \frac{\pi}{2}$$

$$= \frac{\pi}{2} \left[ \frac{1}{2} \sin^2 \theta \cos \phi \cos \frac{\pi}{2} - \sin \theta \cos \phi \sin \frac{\pi}{2} \right] + \dots$$

Now, replacing  $\sin(\theta - \frac{\pi}{2})$  and  $\cos(\theta - \frac{\pi}{2})$  by equations 21 and 22, respectively,



we get:

$$\begin{aligned}\cos j_2 = & \sin \vartheta_0 - \frac{v}{c} \cos \vartheta_0 [\sin \varphi_0 - 2 \sin \mu_0] \\ & + \frac{v^2}{c^2} \left[ -\frac{1}{2} \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \sin \vartheta_0 \cos^2 \varphi_0 \right. \\ & - \frac{1}{2} \sin \vartheta_0 \sin^2 \varphi_0 + 2 \sin \vartheta_0 \sin \varphi_0 \sin \mu_0 \\ & \left. + 2 \cos \vartheta_0 \cos \varphi_0 \sin \mu_0 - 2 \sin \vartheta_0 \sin^2 \mu_0 \right] + \dots\end{aligned}$$

The following expressions will be developed only up to the powers of  $v/c$  to which they are needed when substituted in equations 53 and 54.

$$\sin j_2 \cos j_2 = \sin \vartheta_0 \cos \vartheta_0 + \frac{v}{c} (1 - 2 \cos^2 \vartheta_0) (\sin \varphi_0 - 2 \sin \mu_0) + \dots$$

$$\sin^2 j_2 = \cos^2 \vartheta_0 + 2 \frac{v}{c} \sin \vartheta_0 \cos \vartheta_0 (\sin \varphi_0 - 2 \sin \mu_0) + \dots$$

$$\sin j_2 \cos^2 j_2 = \sin^2 \vartheta_0 \cos \vartheta_0 + \dots$$

$$\sin^2 j_2 \cos j_2 = \sin \vartheta_0 \cos^2 \vartheta_0 + \dots$$

Substituting the appropriate expressions in equation 53, we obtain:

$$\begin{aligned}\sin j_2' = & \cos \vartheta_0 - \frac{v}{c} \sin \vartheta_0 [\sin(2\vartheta_0 + \varphi_0) + 2 \sin \mu_0] \\ & + \frac{v^2}{c^2} \left[ \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \sin \vartheta_0 \sin \varphi_0 \cos \varphi_0 - \frac{1}{2} \cos \vartheta_0 \right. \\ & - 2 \sin(\vartheta_0 + \varphi_0) \sin \varphi_0 + 4 \sin(\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 \sin \varphi_0 \\ & + 4 \sin^2(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 - 2 \sin^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 \\ & + 2 \cos \vartheta_0 \sin \varphi_0 \sin \mu_0 - 2 \sin \vartheta_0 \cos \varphi_0 \sin \mu_0 \\ & + 4 \sin(\vartheta_0 + \varphi_0) \sin \mu_0 - 8 \sin(\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 \sin \mu_0 \\ & \left. - 2 \cos \vartheta_0 \sin^2 \mu_0 \right] + \dots\end{aligned}$$

Eqn. (55)

we get:

$$\begin{aligned} \cos^2 \theta &= \sin^2 \theta - \frac{\sqrt{2}}{2} \cos \theta (\sin \theta - \sqrt{2} \sin^2 \theta) \\ &+ \frac{\sqrt{2}}{2} \left( \frac{1}{2} \sin^2 \theta + \sin \theta \cos \theta + \frac{1}{2} \sin^2 \theta \right) \cos^2 \theta \\ &- \frac{1}{2} \sin^2 \theta \cos^2 \theta + \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta \\ &+ \frac{1}{2} \cos^2 \theta \cos^2 \theta \sin^2 \theta - \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta + \dots \end{aligned}$$

The following expressions will be developed out of the powers

of  $\sin^2 \theta$  to which they are needed, when substituted in equations 12 and 13.

$$\sin^2 \theta \cos^2 \theta = \sin^2 \theta \cos^2 \theta + \frac{\sqrt{2}}{2} (1 - \frac{1}{2} \cos^2 \theta) (\sin^2 \theta - \sqrt{2} \sin^2 \theta) + \dots$$

$$\sin^2 \theta = \cos^2 \theta + \frac{\sqrt{2}}{2} \sin^2 \theta \cos^2 \theta (\sin^2 \theta - \sqrt{2} \sin^2 \theta) + \dots$$

$$\sin^2 \theta \cos^2 \theta = \sin^2 \theta \cos^2 \theta + \dots$$

$$\sin^2 \theta \cos^2 \theta = \sin^2 \theta \cos^2 \theta + \dots$$

Substituting the foregoing expressions in equation 12, we obtain:

$$\begin{aligned} \sin^2 \theta &= \cos^2 \theta - \frac{\sqrt{2}}{2} \sin^2 \theta (\sin^2 \theta + \sqrt{2} \sin^2 \theta) \\ &+ \frac{\sqrt{2}}{2} \left[ \frac{1}{2} \cos^2 \theta + \sin^2 \theta \cos^2 \theta + \sin^2 \theta \cos^2 \theta \cos^2 \theta - \frac{1}{2} \cos^2 \theta \right. \\ &- \frac{1}{2} \sin^2 \theta (\sin^2 \theta + \sqrt{2} \sin^2 \theta) \cos^2 \theta + \frac{1}{2} \sin^2 \theta \cos^2 \theta \cos^2 \theta + \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta \\ &+ \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta - \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta \cos^2 \theta \\ &+ \frac{1}{2} \cos^2 \theta \sin^2 \theta \cos^2 \theta - \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta \cos^2 \theta \\ &+ \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta \cos^2 \theta - \frac{1}{2} \sin^2 \theta \cos^2 \theta \sin^2 \theta \cos^2 \theta \sin^2 \theta \\ &\left. + \dots \right] \sin^2 \theta + \dots \end{aligned}$$

(Eqn. 12)



Also, inserting the proper expressions in equation 54, we get:

$$\begin{aligned}\cos j_2' = & \sin \vartheta_0 + \frac{v}{c} \cos \vartheta_0 [\sin(2\vartheta_0 + \varphi_0) + 2 \sin \mu_0] \\ & + \frac{v^2}{c^2} \left[ -\frac{1}{2} \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \sin \vartheta_0 \cos^2 \varphi_0 \right. \\ & - \frac{1}{2} \sin \vartheta_0 \sin^2 \varphi_0 + 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \varphi_0 \\ & - 4 \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 + 2 \sin \vartheta_0 \sin \varphi_0 \sin \mu_0 \\ & + 2 \cos \vartheta_0 \cos \varphi_0 \sin \mu_0 \\ & \left. - 8 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \mu_0 \right. \\ & \left. - 2 \sin \vartheta_0 \sin^2 \mu_0 \right] + \dots\end{aligned}$$

Eqn. (56)

For later use, it will be convenient to write this equation symbolically, as:

$$\cos j_2' = R + \frac{v}{c} S + \frac{v^2}{c^2} T + \dots \quad \text{Eqn. (57)}$$

where R, S and T are defined as the coefficient of the  $(v/c)^0$ ,  $(v/c)^1$  and  $(v/c)^2$  term, respectively.

#### ANGLE BETWEEN THE TWO FINAL RAYS

Referring to figure I we see that the angle between the ray reflected from M' and the normal to the plate M at the point H is  $i_3$  or  $(\frac{\pi}{2} - \theta' - \sigma - i_2')$ . As a proof of this latter statement, we may sum the angles in the triangle ACH and set them equal to  $\pi$ .

$$\pi = (\vartheta + \vartheta' + \varphi - \psi) + (i_2 + i_2') + (i_3 + \frac{\pi}{2} - \vartheta - \varphi)$$

Also, inserting the proper expressions in relation 26, we get:

$$\begin{aligned} \cos \frac{\gamma}{2} &= \sin \theta_0 \left[ \sin \theta_0 \cos \theta_0 + \frac{1}{2} \sin^2 \theta_0 \right] \\ &+ \frac{\sqrt{2}}{2} \left[ -\frac{1}{2} \sin^2 \theta_0 \cos \theta_0 + \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0 \right] \\ &- \frac{1}{2} \sin \theta_0 \cos^2 \theta_0 + \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0 \\ &- \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0 + \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0 \\ &+ \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0 \\ &- \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0 \\ &- \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0 \end{aligned}$$

and (26)

For later use, it will be convenient to write this equation

symbolically, as:

$$\cos \frac{\gamma}{2} = \sin \theta_0 + \frac{\sqrt{2}}{2} T$$

and (27)

where  $\theta_0$ ,  $B$  and  $T$  are defined as the coefficient of the  $\sin^2 \theta_0$ ,  $\sin \theta_0$  and  $\cos \theta_0$  term, respectively.

### RELATIONSHIP BETWEEN THE TWO CASES

Referring to Figure 1 we see that the angle between the ray reflected from  $S'$  and the normal to the plate is  $\theta_0$  or  $\frac{\gamma}{2}$ . As a result of this latter statement, we may sum the angles in the triangles  $ASN$  and  $ASN'$  and get:

$$\pi = (\theta_0 + \psi - \theta_0) + (\theta_0 + \psi - \theta_0) + \frac{\pi}{2}$$



After replacing  $\psi$  by its equal  $(i_2 - \sigma)$ , we obtain:

$$i_3 = \left( \frac{\pi}{2} - \theta' - \sigma - i_2' \right).$$

Again referring to figure I we observe that the angles  $i_3$  and  $j_2'$  are measured in terms of their angular elevation from parallel base lines--the normals to plate M. Let us then construct a triangle with base N and the angles  $j_2'$  and  $i_3$  as shown in figure VII.

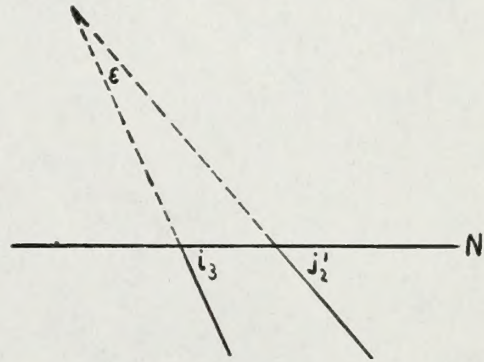


Fig. VII

Unless the two final rays are parallel, there will be a small angle  $\epsilon$  at the summit of the triangle. The two base angles of this triangle are  $(\pi - i_3)$  and  $j_2'$ . Therefore;

$$\begin{aligned} \epsilon &= \pi - (\pi - i_3 + j_2') \\ &= i_3 - j_2' . \end{aligned}$$

Replacing  $i_3$  by its value found above, we get :

$$\epsilon = \left( \frac{\pi}{2} - \theta' - \sigma - i_2' - j_2' \right)$$

Therefore;

$$\sin \epsilon = \cos (\theta' + \sigma + i_2' + j_2')$$

(For details of the derivation of the following equation see appendix A.)

$$\begin{aligned} \sin \epsilon &= 2 \frac{\nu}{c} \sin \mu_0 + \frac{\nu^2}{c^2} \left[ \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 \{ \sin^2(2\vartheta_0 + \varphi_0) - \sin^2(\vartheta_0 + \varphi_0) \} \right. \\ &\quad \left. - 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \right. \\ &\quad \left. + 2 \cos(2\vartheta_0 + \varphi_0) \sin \mu_0 \right] + \dots \quad \text{Eqn. (58)} \end{aligned}$$

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For later use, it will be convenient to write this equation symbolically, as:

$$\sin \epsilon = \frac{v}{c} J' + \frac{v^2}{c^2} K' + \frac{v^3}{c^3} M' + \dots$$

where  $J'$  and  $K'$  represent the coefficients of the first and second order terms respectively. It will be found that the third order term, represented by  $M'$ , will apparently be needed, but upon further development of the problem it will be shown to cancel out completely, hence, it will not be necessary to derive it.

Let us now define  $\epsilon'$  by the equation :

$$\sin \epsilon = \frac{v}{c} \sin \epsilon'$$

Then,

$$\sin \epsilon' = J' + \frac{v}{c} K' + \frac{v^2}{c^2} M' + \dots$$

We shall find that the reciprocal of  $\sin \epsilon'$  will be needed. Thus;

$$\begin{aligned} \frac{1}{\sin \epsilon'} &= \frac{1}{2 \sin \mu_0} \\ &- \frac{v}{c} \frac{1}{4 \sin^3 \mu_0} \left[ \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 \{ \sin^2(2\vartheta_0 + \varphi_0) - \sin^2(\vartheta_0 + \varphi_0) \} \right. \\ &\quad \left. - 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \right. \\ &\quad \left. + 2 \cos(2\vartheta_0 + \varphi_0) \sin \mu_0 \right] \\ &+ \text{second order terms} + \dots \end{aligned} \quad \text{Eqn. (59)}$$

Write this symbolically, as:

$$\frac{1}{\sin \epsilon'} = J + \frac{v}{c} K + \frac{v^2}{c^2} M + \dots \quad \text{Eqn. (60)}$$

where  $J$ ,  $K$  and  $M$  represent the coefficient of the  $(v/c)^0$ ,  $(v/c)^1$  and  $(v/c)^2$  terms, respectively;  $M$  not being actually determined.

For later use, it will be convenient to write this equation

symbolically, as:

$$\sin \epsilon = \frac{v}{c} \left( \frac{v}{c} + \frac{v'}{c} \right) + \frac{v'}{c} M + \dots$$

where  $v'$  and  $M'$  represent the coefficients of the first and second order terms respectively. It will be found that the third order term, represented by  $v''$ , will necessarily be needed, but any further development of the problem it will be found to cancel out completely, hence, it will not be necessary to derive it.

Let us now define  $\epsilon'$  by the equation:

$$\sin \epsilon' = \frac{v}{c} \sin \epsilon$$

Then,

$$\sin \epsilon' = \frac{v}{c} \left( \frac{v}{c} + \frac{v'}{c} \right) + \frac{v'}{c} M + \dots$$

We shall find that the coefficient of  $\sin \epsilon'$  will be needed. Thus,

$$\begin{aligned} \frac{1}{\sin \epsilon'} &= \frac{1}{\frac{v}{c} \sin \epsilon} = \frac{1}{\frac{v}{c} \left[ \frac{v}{c} \left( \frac{v}{c} + \frac{v'}{c} \right) + \frac{v'}{c} M + \dots \right]} \\ &= \frac{1}{\frac{v^2}{c^2} + \frac{v v'}{c^2} + \frac{v'}{c} M + \dots} \\ &= \frac{1}{\frac{v^2}{c^2} \left( 1 + \frac{v'}{v} + \frac{v'}{v} M + \dots \right)} \\ &= \frac{1}{\frac{v^2}{c^2}} \left( 1 - \frac{v'}{v} + \frac{v'}{v} M + \dots \right) \end{aligned}$$

Second order term...

With this symbolically, we

$$\frac{1}{\sin \epsilon'} = \frac{1}{\frac{v^2}{c^2}} \left( 1 + \frac{v'}{v} + \frac{v'}{v} M + \dots \right)$$

where  $\epsilon'$  and  $\epsilon$  represent the inclination of the  $v'$  and  $v$  respectively, and  $M$  and  $M'$  represent the inclination of the  $v'$  and  $v$  respectively.



### LENGTH OF PATH FROM PLATE TO MIRROR M'

Referring to figure I we observe that during the interval of time  $t'_1$  that the ray reflected at the plate M travels from A to C, the mirror M' has moved from E to C. Let  $AC = ct'_1 = D'_1$  and  $EC = AK = vt'_1$ . Then, in triangle ACE, we have, by the sine law: (cf Appendix B)

$$\frac{\sin[\pi - (\theta + \theta' + \phi)]}{D'_1} = \frac{\sin(\theta + \theta' + \phi + \sigma - i_2)}{S'}$$

Solving for  $D'_1$ , we obtain:

$$D'_1 = S' \frac{\sin(\theta + \theta' + \phi)}{\sin(\theta + \theta' + \phi + \sigma - i_2)}$$

Expand denominator in terms of angles  $(\theta + \theta' + \phi)$  and  $(\sigma - i_2)$ . Also, divide numerator and denominator by  $\sin(\theta + \theta' + \phi)$ . This yields:

$$D'_1 = S' \frac{1}{\cos(\sigma - i_2) + \frac{\cos(\theta + \theta' + \phi) \sin(\sigma - i_2)}{\sin(\theta + \theta' + \phi)}}$$

Again, in figure I, referring to triangle ACK, we have by the sine law:

$$\frac{\sin(i_2 - \sigma)}{vt'_1} = \frac{\sin[\pi - (\theta + \theta' + \phi)]}{ct'_1}$$

Solving for  $v/c$  and changing  $\sin(i_2 - \sigma)$  to its equal  $[-\sin(\sigma - i_2)]$  we get:

LENGTH OF PATH FROM PLATE TO PLATE

Referring to Figure 1, we observe that during the interval of time  $t$ , the ray reflected at the glass B travels from A to C, the mirror  $M$  has moved from E to D, but  $AC = CE$ , and  $ED = AE = vt$ . Then, in triangle ADE, we have by the law: (C) Appendix B:

$$\frac{2\pi(\mu-1)g+2\pi\phi}{2c} = \frac{2\pi(\mu-1)g+2\pi\phi}{2c}$$

Solving for  $t$ , we obtain:

$$D' = 2 \frac{2\pi(\mu-1)g+2\pi\phi}{2\pi(\mu-1)g+2\pi\phi}$$

Second denominator is same as before  $(2\pi(\mu-1)g+2\pi\phi)$ . This yields:

$$D' = 2 \frac{1}{\cos^2(\theta) + \frac{\cos(2\theta) \sin(\theta)}{2\pi(\mu-1)g+2\pi\phi}}$$

Again, in Figure 1, referring to triangle ADE, we have by the law:

$$\frac{2\pi(\mu-1)g+2\pi\phi}{2c} = \frac{2\pi(\mu-1)g+2\pi\phi}{2c}$$

Solving for  $\theta$  and changing  $\sin(\theta)$  to the form  $[-\cos(\theta)]$  we get:



$$\frac{v}{c} = -\frac{\sin(\sigma - i_2)}{\sin(\vartheta + \vartheta' + \varphi)}$$

Substitute the latter expression in the equation for  $D'_1$ . Also replace  $i_2$  by its equal given in equation 35. This gives:

$$D'_1 = S' \frac{1}{\sin(\vartheta' + i_1) - \frac{v}{c} \cos(\vartheta + \vartheta' + \varphi)}$$

Replacing  $\sin(\vartheta' + i_1)$  and  $\cos(\vartheta + \vartheta' + \varphi)$  by equations 31 and 8 respectively, results in:

$$D'_1 = S' \frac{1}{1 - \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) - \frac{v^2}{c^2} \left[ \frac{1}{2} \sin^2 \varphi_0 - 2 \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \{ \sin \varphi_0 - \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \right]}$$

or

$$D'_1 = S' \frac{1}{1 - \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) - \frac{v^2}{c^2} \left[ \frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \right]}$$

Expand this expression in terms of powers of  $v/c$  and drop powers greater than the second.

$$D'_1 = S' \left[ 1 + \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) + \frac{v^2}{c^2} \left\{ \cos^2(2\vartheta_0 + \varphi_0) + \frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \right\} + \dots \right]$$

Applying the Lorentz transformation equation 19 to the above, we obtain:

$$D'_1 = S'_0 \left[ 1 + \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) + \frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \right\} + \dots \right]$$

Expanding the term  $\frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0)$  in terms of angles  $(\vartheta_0 + \varphi_0)$  and  $\vartheta_0$  we find that

$$\frac{V}{C} = \frac{\sin(\theta - \theta')}{\sin(\theta + \theta')}$$

Substituting the latter expression in the equation for  $D'$ , also replacing  $\frac{V}{C}$

its usual value in equation 31. This gives:

$$D' = \frac{1}{\sin(\theta + \theta') - \frac{V}{C} \cos(\theta + \theta')}$$

Replacing  $\sin(\theta + \theta')$  and  $\cos(\theta + \theta')$  by equations 31 and 3 respectively,

results in:

$$D' = \frac{1}{1 - \frac{V}{C} \cos(\theta + \theta') - \frac{V^2}{C^2} [\frac{1}{2} \sin^2 \theta' - 2 \sin \theta \sin \theta' \cos \theta + \sin^2 \theta \cos^2 \theta]}$$

or

$$D' = \frac{1}{1 - \frac{V}{C} \cos(\theta + \theta') - \frac{V^2}{C^2} [\frac{1}{2} \sin^2 \theta' + 2 \sin \theta \sin \theta' \cos \theta + \sin^2 \theta \cos^2 \theta]}$$

Express this expression in terms of powers of  $V/C$  and drop powers higher than

the second.

$$D' = 2' [1 + \frac{V}{C} \cos(\theta + \theta')]$$

$$+ \frac{V^2}{C^2} [\cos^2(\theta + \theta') + \frac{1}{2} \sin^2 \theta' + 2 \sin \theta \sin \theta' \cos \theta + \sin^2 \theta \cos^2 \theta]$$

Replace the lower terms in the bracket in the above, in order:

$$D' = 2' [1 + \frac{V}{C} \cos(\theta + \theta')]$$

$$+ \frac{V^2}{C^2} [\cos^2(\theta + \theta') + \frac{1}{2} \sin^2 \theta' + 2 \sin \theta \sin \theta' \cos \theta + \sin^2 \theta \cos^2 \theta]$$

Expressing the term  $\cos^2(\theta + \theta')$  in terms of angles  $\theta$  and  $\theta'$ , we find that



the second order term reduces to  $\frac{1}{2}$ . Thus, we get as the final expression for  $D_1'$  :

$$D_1' = S_0' \left[ 1 + \frac{v}{c} \cos(2\theta_0 + \varphi_0) + \frac{1}{2} \frac{v^2}{c^2} + \dots \right] \quad \text{Eqn. (61)}$$

#### LENGTH OF PATH FROM MIRROR M' BACK TO PLATE

Referring to figure I we observe that in the interval of time  $t_2'$  that the ray reflected at the mirror M' travels from C to H, the mirror M' has moved a distance equal to KH. Let  $CH = ct_2' = D_2'$  and  $KH = vt_2'$ . Then, in triangle CHK, we have, by the sine law: (cf. Appendix B)

$$\frac{\sin(\theta + \theta' + \varphi)}{D_2'} = \frac{\sin(\theta + \theta' + \varphi + \sigma + i_2')}{S'}$$

As a proof of this latter statement observe that

$$\text{angle CHK} = i_3 + \frac{\pi}{2} - \theta - \phi$$

Replace  $i_3$  by its equal  $(\frac{\pi}{2} - \theta' - \sigma - i_2')$ . Then:

$$\text{angle CHK} = \pi - \theta' - \sigma - i_2' - \theta - \phi$$

The sine of the angle  $\text{CHK} = \sin[\pi - (\theta + \theta' + \phi + \sigma + i_2')]$

$$= \sin(\theta + \theta' + \varphi + \sigma + i_2')$$

Return to the equation involving  $D_2'$  and solve for  $D_2'$ .

The second order term reduces to 0. Then, we get as the final expression

for  $D'$ :

$$D' = z_0' \left[ \left( 1 + \frac{v}{c} \cos(\theta + \phi) \right) + \frac{1}{2} \left( \frac{v}{c} \right)^2 \right] \quad \text{Eq. (6)}$$

### LENGTH OF PATH FROM SOURCE TO RECEIVER

Referring to Figure 1 we observe that in the interval of time  $t_0'$  that the ray reflected at the mirror  $M'$  travels from  $E$  to  $B'$ , the mirror  $B'$  has moved a distance equal to  $Kt_0'$ . Let  $OB = ct_0'$  and  $Kt_0 = vt_0'$ . Then, in triangle  $OBK$ , we have, by the sine law: (cf Appendix D)

$$\frac{\sin(\theta + \phi)}{D'} = \frac{\sin(\theta + \phi + \psi)}{2'}$$

As a result of this relation it follows that

$$\sin \psi = \frac{v}{c} \sin(\theta + \phi)$$

Replace  $\psi$  by the value  $\frac{\pi}{2} - (\theta' - \phi')$ . Then:

$$\sin(\theta' - \phi') = \frac{v}{c} \sin(\theta + \phi)$$

The sine of the angle  $OBK = \sin(\theta + \phi + \psi) = \sin(\theta + \phi + \frac{\pi}{2} - (\theta' - \phi'))$

$$= \sin(\theta + \phi + \frac{\pi}{2} - \theta' + \phi')$$

Return to the equation involving  $D'$  and solve for  $D'$ .



$$D_2' = S' \frac{\sin(\vartheta + \vartheta' + \varphi)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')}$$

Expand denominator in terms of angles  $(\vartheta + \vartheta' + \varphi)$  and  $(\sigma + i_2')$ ; and divide numerator and denominator by  $\sin(\vartheta + \vartheta' + \varphi)$ . This yields:

$$D_2' = S' \frac{1}{\cos(\sigma + i_2') + \frac{\cos(\vartheta + \vartheta' + \varphi) \sin(\sigma + i_2')}{\sin(\vartheta + \vartheta' + \varphi)}}$$

Again, in figure I, referring to triangle CHK we have by the sine law:

$$\frac{vt_2'}{ct_2'} = \frac{\sin(i_1' + \sigma)}{\sin(\vartheta + \vartheta' + \varphi)}$$

Substitute the latter expression in the equation for  $D_2'$ . This gives:

$$D_2' = S' \frac{1}{\cos(\sigma + i_2') + \frac{v}{c} \cos(\vartheta + \vartheta' + \varphi)}$$

Expand  $\cos(\sigma + i_2')$  and replace  $\cos \sigma$  and  $\sin \sigma$  by equations 15 and 16 respectively. This results in the equation:

$$D_2' = S' \frac{1}{\cos i_2' + \frac{v}{c} \cos(\vartheta + \vartheta' + \varphi) - \frac{v^2}{c^2} \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \sin i_2'}$$

Substituting equations 8, 39 and 40 and dropping powers of  $v/c$  greater than the second, gives:

$$D_2' = S' \frac{1}{1 + \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) - \frac{v^2}{c^2} [\frac{1}{2} \sin^2 \varphi_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0]}$$

Expand in terms of powers of  $v/c$ .

$$D_1' = z' \frac{\sin(\theta + \phi)}{\sin(\theta + \phi + \psi)}$$

Expand denominator in terms of angles  $(\theta + \phi)$  and  $(\phi + \psi)$ :

Divide numerator and denominator by  $\sin(\theta + \phi)$ . This gives:

$$D_1' = z' \frac{1}{\cos(\phi + \psi) + \frac{\cos(\theta + \phi + \psi)}{\sin(\theta + \phi)}}$$

Again, in Figure 1, referring to triangle EFG we have by the sine law:

$$\frac{EF}{\sin \phi} = \frac{FG}{\sin(\theta + \phi)}$$

Substitute the latter expression in the equation for  $D_1'$ . This gives:

$$D_1' = z' \frac{1}{\cos(\phi + \psi) + \frac{\sin \phi}{\sin(\theta + \phi)}}$$

Expand  $\cos(\phi + \psi)$  and replace  $\sin \phi$  and  $\sin(\theta + \phi)$  by equations 19 and 18, respectively:

This results in the equation:

$$D_1' = z' \frac{1}{\cos \phi' + \frac{\sin \phi'}{\sin(\theta + \phi)} - \frac{\sin \phi'}{\sin(\theta + \phi)} \cos(\theta + \phi) \cos \phi'}$$

Substituting equations 2, 3, and 4 and dropping powers of  $\phi'$  greater than

the second, gives:

$$D_1' = z' \frac{1}{1 + \frac{1}{2} \cos(\theta + \phi) - \frac{1}{2} \sin^2 \theta + \frac{1}{2} \sin^2 \theta \cos(\theta + \phi) \cos \phi'}$$

Expand in terms of powers of  $\phi'$ .



$$D_2' = S' \left[ 1 - \frac{v}{c} \cos(2\theta_0 + \phi_0) + \frac{v^2}{c^2} \left\{ \cos^2(2\theta_0 + \phi_0) + \frac{1}{2} \sin^2 \phi_0 + 2 \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin \theta_0 \cos \theta_0 \right\} + \dots \right].$$

Applying the Lorentz transformation equation 19 to the above, we obtain:

$$\text{the } D_2' = S_0' \left[ 1 - \frac{v}{c} \cos(2\theta_0 + \phi_0) + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2(2\theta_0 + \phi_0) + \frac{1}{2} \sin^2 \phi_0 + 2 \sin(\theta_0 + \phi_0) \cos(\theta_0 + \phi_0) \sin \theta_0 \cos \theta_0 \right\} + \dots \right]$$

Reducing the second order term as we did in the case of  $D_1'$  we obtain as the final expression for  $D_2'$  :

$$D_2' = S_0' \left[ 1 - \frac{v}{c} \cos(2\theta_0 + \phi_0) + \frac{1}{2} \frac{v^2}{c^2} + \dots \right] \quad \text{Eqn. (62)}$$

It will be observed, that except for the sign of the  $v/c$  terms the two distances  $D_1'$  and  $D_2'$  are equal.

#### LENGTH OF PATH FROM PLATE M TO MIRROR M''

In figure I construct the line AQ parallel to  $S''$ . This line will also equal  $S''$  since they are both bounded by the same parallel lines. During the interval of time that the ray has traveled from A to P the mirror  $M''$  has moved from Q to P. Let  $AP = ct'' = D_1''$  and  $QP = vt''$ . Then, in triangle APQ we have by the sine law:

$$\frac{\sin(\varphi - \alpha)}{S''} = \frac{\sin(\pi - \varphi)}{D_1''}$$

Solving for  $D_1''$  we obtain:

$$D'_1 = 2 \left[ 1 - \frac{\sqrt{2}}{2} \cos(\theta_1 + \phi_1) \right]$$

$$+ \frac{\sqrt{2}}{2} \left\{ \cos(\theta_1 + \phi_1) + \frac{1}{2} \sin(\theta_1 + \phi_1) \cos(\theta_1 + \phi_1) + \dots \right\}$$

Applying the Lorentz transformation equation to the above, we obtain:

$$-D'_1 = 2 \left[ 1 - \frac{\sqrt{2}}{2} \cos(\theta_1 + \phi_1) \right]$$

$$+ \frac{\sqrt{2}}{2} \left\{ \cos(\theta_1 + \phi_1) + \frac{1}{2} \sin(\theta_1 + \phi_1) \cos(\theta_1 + \phi_1) + \dots \right\}$$

Reducing the second order terms as we did in the case of  $D_1$  we obtain in the

final expression for  $D'_1$ :

$$D'_1 = 2 \left[ 1 - \frac{\sqrt{2}}{2} \cos(\theta_1 + \phi_1) + \frac{1}{2} \frac{v^2}{c^2} + \dots \right]$$

Eqn. (62)

It will be observed, that except for the sign of the  $v^2/c^2$  term

the two distances  $D_1$  and  $D'_1$  are equal.

### LENGTH OF LINE PLATE M TO VIEWER IN

In Figure 2, consider the line M parallel to  $E''$ . This line will

also appear  $E''$  since it is parallel to the same parallel plane.

During the interval of time that the ray has traveled from A to E the mirror

$E'$  has moved from Q to P. Let  $AP = v_1 t_1$  and  $QP = v_2 t_1$ . Then, in triangle

$APQ$  we have by the sine law:

$$\frac{\sin(\theta - \phi)}{v_1} = \frac{\sin(\theta - \phi)}{v_2}$$

Noting that  $v_1$  and  $v_2$  are equal,



$$D_1'' = S'' \frac{\sin \phi}{\sin(\phi - \alpha)}$$

Expand the denominator.

$$D_1'' = S'' \frac{\sin \phi}{\sin \phi \cos \alpha - \cos \phi \sin \alpha}$$

Substitute equations 3, 4, 24 and 25 and divide numerator and denominator by expression for  $\sin \phi$ . This yields,

$$D_1'' = S'' \frac{1}{1 - \frac{v}{c} \cos \phi_0 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \phi_0}$$

Expanding in terms of powers of  $v/c$  we obtain

$$D_1'' = S'' \left[ 1 + \frac{v}{c} \cos \phi_0 + \frac{1}{2} \frac{v^2}{c^2} (1 + \cos^2 \phi_0) + \dots \right]$$

Applying the Lorentz transformation equation 20 we obtain as the final equation for  $D_1''$  :

$$D_1'' = S_0'' \left[ 1 + \frac{v}{c} \cos \phi_0 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right] \quad \text{Eqn. (63)}$$

#### LENGTH OF PATH FROM MIRROR M'' BACK TO PLATE M

In figure I construct FP parallel to  $S''$ . This line, will also equal  $S''$  since they are both bounded by the same parallel lines. During the interval of time that the ray of light has traveled from P to L the plate M has travelled a distance FR, Let  $PL = ct_2'' = D_1''$  and  $FR = vt_2''$ . Let B be the point of intersection of the plate M and the line FP at the time  $t_2''$ .

$$D'' = S'' \frac{\sin \phi}{2 \cos(\phi - \theta)}$$

Expand the denominator,

$$D'' = S'' \frac{\sin \phi}{\sin \phi \cos \theta - \cos \phi \sin \theta}$$

Substitute equations 3, 4, 24 and 25 and divide numerator and denominator by  $\sin \phi$ . This yields,

$$D'' = S'' \frac{1}{1 - \frac{\cos \phi}{\sin \phi} \cos \theta + \frac{\sin \phi}{\sin \phi} \sin \theta}$$

Expanding in terms of powers of  $\theta$  we obtain

$$D'' = S'' \left[ 1 + \frac{\cos \theta}{\sin \phi} + \frac{1}{2} \frac{\cos^2 \theta}{\sin^2 \phi} + \dots \right]$$

Applying the binomial expansion equation 25 we obtain in the final equation for  $D''$  :

$$D'' = S'' \left[ 1 + \frac{\cos \theta}{\sin \phi} + \frac{1}{2} \frac{\cos^2 \theta}{\sin^2 \phi} + \dots \right] \quad \text{eqn. (22)}$$

### LENGTH OF PATH FROM MIRROR M BACK TO PLATE N

In Figure 1 constant  $FE$  is equal to  $S''$ . This line, which also equals  $EN$ , since they are both bounded by the same parallel lines. During the interval of time that the ray of light has traveled from  $P$  to  $L$  the plate  $N$  has traveled a distance  $EN$ . Let  $EC = S'' \sin \theta$  and  $EN = S'' \cos \theta$ . Let  $S$  be the point of intersection of the plate  $N$  and the line  $FE$  at the time  $t_2$ .



Now,  $FB + BP = S''$

The distance that the plate M has moved in the interval of time  $t_2''$  is FN or  $vt_2'' \cos[\frac{\pi}{2} - (\theta + \phi)]$  or  $vt_2'' \sin(\theta + \phi)$ . The latter is the perpendicular distance from F to M at the end of the time interval  $t_2''$ . Therefore, in triangle FNB

$$FB = \frac{vt_2'' \sin(\theta + \phi)}{\sin \theta}.$$

Substituting this value of FB in the equation above, we obtain:

$$BP = S'' - \frac{vt_2'' \sin(\theta + \phi)}{\sin \theta}.$$

In triangle BPL, we have, by the sine law:

$$\frac{\sin(\pi - \theta)}{ct_2''} = \frac{\sin(\frac{\pi}{2} - j_2)}{BP}.$$

Replacing BP by its value just obtained and solving for  $t_2''$ , we obtain:

$$t_2'' = S'' \frac{\sin \theta}{c \cos j_2 + v \sin(\theta + \phi)}.$$

Multiplying both sides of the equation by  $c$  and replacing  $ct_2''$  by its equal,  $D_2''$ , yields:

$$D_2'' = S'' \frac{\sin \theta}{\cos j_2 + \frac{v}{c} \sin(\theta + \phi)}.$$

Substituting equation 5 for  $\sin(\theta + \phi)$  and the equation given in page 51, for  $\cos j_2$  we obtain as the denominator of the expression for  $D_2''$ :

$$EB + BP = 2''$$

The distance from the plate to the lens is the interval of time  $t_1''$  to  $W$  or  $t_1'' \cos(\frac{\pi}{2} - \theta) = t_1'' \sin \theta$ . The latter is the perpendicular distance from  $E$  to  $B$  at the end of the interval  $t_1''$ . Therefore, in triangle  $EBP$

$$EB = \frac{vt_1'' \sin(\frac{\pi}{2} - \theta)}{\sin \theta}$$

Substituting the value of  $EB$  in the equation above, we obtain:

$$BP = 2'' - \frac{vt_1'' \sin(\frac{\pi}{2} - \theta)}{\sin \theta}$$

In triangle  $BP$ , we have, by the sine law:

$$\frac{\sin(\theta - \theta)}{BP} = \frac{\sin(\frac{\pi}{2} - \theta)}{t_1''}$$

Replacing  $BP$  by its value from equation and solving for  $t_1''$ , we obtain:

$$t_1'' = 2'' \frac{\sin \theta}{c \cos \theta + v \sin(\theta + \theta)}$$

Multiplying both sides of the equation by  $c$  and replacing  $c t_1''$  by the value  $D_1''$

we obtain:

$$D_1'' = 2'' \frac{\sin \theta}{\cos \theta + \frac{v}{c} \sin(\theta + \theta)}$$

Substituting equation 5 for  $\sin(\theta + \theta)$  and the equation given in page 21, for

we obtain as the denominator of the expression for  $D_1''$ :



$$\begin{aligned}\cos j_2 + \frac{v}{c} \sin(4\phi) = & \sin \vartheta_0 + \frac{v}{c} [\sin \vartheta_0 \cos \phi_0 + 2 \cos \vartheta_0 \sin \mu_0] \\ & + \frac{v^2}{c^2} [-\frac{1}{2} \sin^2(\vartheta_0 + \phi_0) \sin \vartheta_0 + \frac{1}{2} \sin \vartheta_0 \cos^2 \phi_0 \\ & - \frac{1}{2} \sin \vartheta_0 \sin^2 \phi_0 + 2 \sin \vartheta_0 \sin \phi_0 \sin \mu_0 \\ & + 2 \cos \vartheta_0 \cos \phi_0 \sin \mu_0 - 2 \sin \vartheta_0 \sin^2 \mu_0] + \dots\end{aligned}$$

Substitute the latter expression in equation for  $D_2''$  and expand in terms of  $v/c$ . This gives:

$$\begin{aligned}D_2'' = S'' \sin \vartheta \left[ \frac{1}{\sin \vartheta_0} - \frac{v}{c} \frac{1}{\sin^2 \vartheta_0} \{ \sin \vartheta_0 \cos \phi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} \right. \\ \left. + \frac{v^2}{c^2} \frac{1}{\sin^3 \vartheta_0} \{ \sin^2 \vartheta_0 \cos^2 \phi_0 + 4 \cos^2 \vartheta_0 \sin^2 \mu_0 \right. \\ \left. + 2 \sin \vartheta_0 \cos \vartheta_0 \cos \phi_0 \sin \mu_0 + \frac{1}{2} \sin^2 \vartheta_0 \sin^2(\vartheta_0 + \phi_0) \right. \\ \left. + \frac{1}{2} \sin^2 \vartheta_0 \sin^2 \phi_0 - \frac{1}{2} \sin^2 \vartheta_0 \cos^2 \phi_0 \right. \\ \left. + 2 \sin^2 \vartheta_0 \sin^2 \mu_0 - 2 \sin \vartheta_0 \cos \vartheta_0 \cos \phi_0 \sin \mu_0 \right. \\ \left. - 2 \sin^2 \vartheta_0 \sin \phi_0 \sin \mu_0 \} + \dots \right],\end{aligned}$$

which reduces to:

$$\begin{aligned}D_2'' = S'' \sin \vartheta \left[ \frac{1}{\sin \vartheta_0} - \frac{v}{c} \frac{1}{\sin^2 \vartheta_0} \{ \sin \vartheta_0 \cos \phi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} \right. \\ \left. + \frac{v^2}{c^2} \frac{1}{\sin^3 \vartheta_0} \left\{ \frac{1}{2} \sin^2 \vartheta_0 [1 + \sin^2(\vartheta_0 + \phi_0)] \right. \right. \\ \left. \left. + 2 \cos(\vartheta_0 + \phi_0) \sin \vartheta_0 \sin \mu_0 \right. \right. \\ \left. \left. + 2(1 + \cos^2 \vartheta_0) \sin^2 \mu_0 \right\} + \dots \right].\end{aligned}$$

Replacing  $\sin \theta$  and  $S''$  by equations 9 and 20 respectively, we obtain as the final expression for  $D_2''$  :

$$\begin{aligned}D_2'' = S_0'' \left[ 1 - \frac{v}{c} \frac{1}{\sin \vartheta_0} \{ \sin \vartheta_0 \cos \phi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} \right. \\ \left. + \frac{v^2}{c^2} \frac{1}{\sin^2 \vartheta_0} \left\{ \frac{1}{2} \sin^2 \vartheta_0 + 2 \cos(\vartheta_0 + \phi_0) \sin \mu_0 \sin \vartheta_0 \right. \right. \\ \left. \left. + 2(1 + \cos^2 \vartheta_0) \sin^2 \mu_0 \right\} + \dots \right].\end{aligned}$$

Eqn. (64)

It will be observed that if  $\mu_0$  were set equal to zero equation 64 would reduce to equation 63 with the single exception of the different sign in the  $v/c$  terms. Also, there is a close resemblance of equations 61 and 62 with equations 63 and 64.

$$\cos \frac{1}{2} \pi + \frac{\sqrt{2}}{2} \sin \frac{1}{2} \pi = \sin \frac{1}{2} \pi + \frac{\sqrt{2}}{2} \cos \frac{1}{2} \pi$$

$$+ \frac{\sqrt{2}}{2} \left[ -\frac{1}{2} \sin^2 \theta + \cos^2 \theta \right] + \frac{1}{2} \sin \theta \cos \theta$$

$$- \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \sin \theta \cos \theta$$

$$+ 2 \cos \theta \cos \theta \sin \theta - 2 \sin \theta \sin \theta + \dots$$

Substitute the first expression in equation (1) and expand in terms of

$\theta$ . This gives:

$$D_1'' = 2'' \sin \theta \left[ \frac{1}{\sin \theta} - \frac{1}{\cos \theta} \right] + \frac{1}{\cos \theta} \left[ \sin^2 \theta \cos \theta + 2 \cos \theta \sin \theta \right]$$

$$+ \frac{\sqrt{2}}{2} \left[ \sin^2 \theta \cos \theta + 2 \cos \theta \sin \theta \right]$$

$$+ 2 \sin \theta \cos \theta \sin \theta + \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \sin \theta \cos \theta$$

$$+ \frac{1}{2} \sin \theta \cos \theta - \frac{1}{2} \sin \theta \cos \theta$$

$$+ 2 \sin \theta \cos \theta \sin \theta - 2 \sin \theta \cos \theta \sin \theta$$

$$- 2 \sin \theta \cos \theta \sin \theta + \dots$$

which reduces to:

$$D_1'' = 2'' \sin \theta \left[ \frac{1}{\sin \theta} - \frac{1}{\cos \theta} \right] + \frac{1}{\cos \theta} \left[ \sin^2 \theta \cos \theta + 2 \cos \theta \sin \theta \right]$$

$$+ \frac{\sqrt{2}}{2} \left[ \sin^2 \theta \cos \theta + 2 \cos \theta \sin \theta \right]$$

$$+ 2 \cos \theta \sin \theta \sin \theta$$

$$+ 2 (1 + \cos^2 \theta) \sin \theta + \dots$$

Replacing  $\sin \theta$  and  $\cos \theta$  by equations (2) and (3) respectively, we obtain as the

final expression for  $D_1''$ :

$$D_1'' = 2'' \left[ 1 - \frac{\sqrt{2}}{2} \sin \theta \right] + \frac{1}{\cos \theta} \left[ \sin^2 \theta \cos \theta + 2 \cos \theta \sin \theta \right]$$

$$+ \frac{\sqrt{2}}{2} \left[ \sin^2 \theta \cos \theta + 2 \cos \theta \sin \theta \right]$$

$$+ 2 (1 + \cos^2 \theta) \sin \theta + \dots$$

and (3)

It will be observed that the terms not equal to zero equation (1) would reduce to equation (2) with the slight exception of the different sign in the  $\sqrt{2}$  term. Also, there is a close resemblance of equations (2) and (3) with equations (4) and (5).



### WAVE NUMBERS OF THE RAYS ALONG THE VARIOUS PATHS

Equation 23, derived earlier in this paper, gives the relationship between the reciprocal of the wave length of the light reflected from a moving mirror and the reciprocal of the wave length of the incident light. In order that we may find the reciprocal wave lengths or wave numbers as measured by the moving observer, we must apply the Lorentz transformations to equation 23.

### WAVE NUMBER OF THE LIGHT ALONG PATH D<sub>1</sub>

To find the wave number of the light along path D<sub>1</sub> apply equation 23. Let  $1/\lambda$  and  $1/\lambda_1$  be the wave numbers of the incident and reflected light, respectively. Referring to equation 27, we see that  $v/c \sin \phi$  must be replaced by  $-v/c \sin(\theta + \phi)$ . Therefore:

$$\frac{1}{\lambda_1} = \frac{1}{\lambda} \left[ 1 - 2 \frac{v}{c} \sin(\theta + \phi) \cos i_1 + 2 \frac{v^2}{c^2} \sin^2(\theta + \phi) + \dots \right].$$

Replace  $\cos i_1$  by its value found on page 38. This gives:

$$\frac{1}{\lambda_1} = \frac{1}{\lambda} \left[ 1 - 2 \frac{v}{c} \sin(\theta + \phi) \sin \theta + 2 \frac{v^2}{c^2} \sin(\theta + \phi) \{ \sin(\theta + \phi) - \cos \theta \sin \phi \} + \dots \right].$$

Simplify the second order term and apply equations 3, 5 and 9 .

$$\begin{aligned} \frac{1}{\lambda_1} = \frac{1}{\lambda} & \left[ 1 - 2 \frac{v}{c} \sin(\theta_0 + \phi_0) \sin \theta_0 \right. \\ & \left. + 2 \frac{v^2}{c^2} \sin(\theta_0 + \phi_0) \sin \theta_0 \cos \phi_0 + \dots \right]. \end{aligned}$$

Eqn. (65)

# WAVE NUMBER OF THE LIGHT ALONG THE VARIOUS PATHS

Equation 27, derived earlier in this paper, gives the relationship

between the reciprocal of the wave length of the light reflected from a moving mirror and the reciprocal of the wave length of the incident light. In order that we may find the reciprocal wave lengths or wave numbers as measured by the moving observer, we must apply the Lorentz transformation to equation 27.

## WAVE NUMBER OF THE LIGHT ALONG PATH D

To find the wave number of the light along path D, apply equation 27. Let  $\lambda$  and  $\lambda'$  be the wave numbers of the incident and reflected light, respectively. Referring to equation 27, we see that  $v/c \sin \theta$  must be replaced by  $-v/c \sin(\theta + \phi)$ . Therefore:

$$\frac{1}{\lambda'} = \frac{1}{\lambda} \left[ 1 - \frac{v}{c} \sin(\theta + \phi) \cos \theta + \frac{v^2}{2c^2} \sin^2(\theta + \phi) \right] \quad [1]$$

Replace cos  $\phi$  by its value found in eqn 28. This gives:

$$\frac{1}{\lambda'} = \frac{1}{\lambda} \left[ 1 - \frac{v}{c} \sin(\theta + \phi) \sin \theta + \frac{v^2}{2c^2} \sin^2(\theta + \phi) - \cos^2 \theta \frac{v^2}{2c^2} \right] \quad [2]$$

Simplify the second order term and apply equations 2, 3 and 4.

$$\frac{1}{\lambda'} = \frac{1}{\lambda} \left[ 1 - \frac{v}{c} \sin \theta \sin(\theta + \phi) \right] + \frac{v^2}{2c^2} \sin^2 \theta \cos^2 \theta + \dots$$



The wave length measured by the moving observer is obtained by applying the Lorentz transformation to  $\lambda$  and  $\lambda'_1$ . From figure I we see that the ray of light which is incident on the plate M makes an angle  $(\phi - \alpha)$  with the direction of the velocity. Therefore, applying equation 1, we get:

$$\lambda = \lambda_0 \left[ 1 - \frac{v^2}{c^2} \cos^2(\phi - \alpha) \right]^{\frac{1}{2}}$$

where  $\lambda_0$  is the wave length of the incident light as measured by the moving observer. Expand  $\cos^2(\phi - \alpha)$  and substitute equations 3, 4, 24 and 25. Since  $\cos^2(\phi - \alpha)$  is multiplied by  $(v/c)^2$ , it is equal to  $\cos^2\phi_0$  to the order of approximation that it is needed. Therefore, taking the reciprocal of the last equation and expanding, we obtain:

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2\phi_0 + \dots \right]. \quad \text{Eqn. (66)}$$

In equation 65 replace  $1/\lambda$  by the latter expression, this yields:

$$\begin{aligned} \frac{1}{\lambda'_1} = \frac{1}{\lambda_0} & \left[ 1 - 2 \frac{v}{c} \sin(\vartheta_0 + \phi_0) \sin\vartheta_0 \right. \\ & \left. + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2\phi_0 + 2 \sin(\vartheta_0 + \phi_0) \sin\vartheta_0 \cos\phi_0 \right\} + \dots \right]. \end{aligned} \quad \text{Eqn. (67)}$$

It is now necessary to apply the Lorentz transformation to  $1/\lambda'_1$  itself. Examining figure I we see that the angle between  $D'_1$  and the direction of the velocity is  $(\theta + \theta' + \phi - \psi)$ . Replacing  $\psi$  by its equal  $(i'_1 + \theta' - \frac{\pi}{2})$ , we find this angle to be  $(\frac{\pi}{2} + \theta + \phi - i'_1)$ . The cosine of this latter angle is found to be  $\cos(2\theta_0 + \phi_0)$  to the order of approximation to which it is needed. Thus:

$$\frac{1}{\lambda'_1} = \frac{1}{\lambda'_1} \left[ 1 - \frac{v^2}{c^2} \cos^2(2\theta_0 + \phi_0) \right]^{-\frac{1}{2}}$$

Expanding, we obtain:

The wave length measured by the moving observer is obtained by applying the Lorentz transformation to  $\lambda$  and  $\lambda'$ . From Figure 1 we see that the ray of light which is incident on the plate  $P$  makes an angle  $(\theta - \theta')$  with the direction of the velocity. Therefore, applying equation 1, we get:

$$\lambda' = \lambda_0 \left[ \frac{1 - \cos(\theta - \theta')}{1 - \beta^2} \right]^{1/2}$$

where  $\lambda_0$  is the wave length of the incident light as measured by the moving observer. Expanding  $\cos(\theta - \theta')$  and substituting equations 2, 4, 12 and 13, since  $\cos(\theta - \theta') = 1$  is multiplied by  $(v/v_0)^2$ , it is equal to  $\cos^2 \theta_0$  to the order of approximation that is needed. Therefore, taking the reciprocal of the last equation and expanding, we obtain:

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0} \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2 \theta_0 + \dots \right] \quad \text{Eq. (3)}$$

In equation 3 replace  $\lambda_0$  by the Lorentz contraction, this yields:

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0} \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \theta_0 + \dots \right] + \frac{1}{2} \frac{v^2}{c^2} \cos^2 \theta_0 + \dots$$

Eq. (4)

It is now necessary to apply the Lorentz transformation to  $\lambda'/\lambda_0$ . From Figure 1 we see that the angle between  $\lambda'$  and the direction of the velocity is  $(\theta' - \theta) = \theta' - \theta$ . Applying  $\psi$  by the same method as before, we find that  $\theta' = \theta - \theta'$ . The value of  $\theta'$  for angle  $\theta$  is found to be  $\cos(\theta' - \theta) = \cos(\theta - \theta')$  to the order of approximation in which it is needed. Then:

$$\frac{\lambda'}{\lambda_0} = \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \cos^2(\theta - \theta') + \dots \right]^{1/2}$$

Therefore, we obtain:



$$\frac{1}{\lambda'_1} = \frac{1}{\lambda_{10}} \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) + \dots \right] \quad \text{Eqn. (68)}$$

Substituting this latter expression for  $1/\lambda'_1$  in equation 67 we obtain as the final equation of the wave number of the light along path  $D'_1$ :

$$\frac{1}{\lambda'_{10}} = \frac{1}{\lambda_0} \left[ 1 - 2 \frac{v}{c} \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2 \varphi_0 - \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) + 2 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \varphi_0 \right\} + \dots \right] \quad \text{Eqn. (69)}$$

#### WAVE NUMBER OF THE LIGHT ALONG PATH $D'_2$ AND $D'_3$

Let  $1/\lambda'_2$  be the wave number of the light along path  $D'_2$ . This will also be the wave number of the light along the path HT or  $D'_3$ . The path  $D'_3$  will be discussed in detail in a later section. Applying equation 23 and making the same substitution for  $v/c \sin \phi$  as was made in equation 33 we obtain:

$$\frac{1}{\lambda'_2} = \frac{1}{\lambda'_1} \left[ 1 - 2 \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) \cos i_2 + 2 \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) + \dots \right]$$

Substitute equation 37 for  $\cos i_2$ . This yields:

$$\frac{1}{\lambda'_2} = \frac{1}{\lambda'_1} \left[ 1 - 2 \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) + 2 \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) + \dots \right]$$

Replace  $1/\lambda'_1$  by equation 68.

$$\frac{1}{\lambda'_2} = \frac{1}{\lambda_{10}} \left[ 1 - 2 \frac{v}{c} \cos(2\vartheta_0 + \varphi_0) + \frac{5}{2} \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) + \dots \right]$$

Now, substitute equation 69 for  $1/\lambda'_{10}$ . This yields:

$$\begin{aligned} \frac{1}{\lambda'_2} = \frac{1}{\lambda_0} & \left[ 1 - 2 \frac{v}{c} \{ \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \cos(2\vartheta_0 + \varphi_0) \} \right. \\ & + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2 \varphi_0 - \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) + 2 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \varphi_0 \right. \\ & \left. \left. + 4 \cos(2\vartheta_0 + \varphi_0) \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{5}{2} \cos^2(2\vartheta_0 + \varphi_0) \right\} + \dots \right] \end{aligned}$$

Eq. (22)

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_0} \left[ 1 + \frac{v}{c} \cos(\theta_0 + \phi_0) + \dots \right]$$

Substituting this latter expression for  $\frac{1}{\lambda_0}$  in equation (21) we obtain as the final equation of the wave number of the light along path B:

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \sin(\theta_0 + \phi_0) \sin \theta_0 \right]$$

$$+ \frac{v^2}{c^2} \left[ \frac{1}{2} \cos^2 \theta_0 - \frac{1}{2} \cos^2(\theta_0 + \phi_0) + \sin(\theta_0 + \phi_0) \sin \theta_0 \cos \theta_0 \right] \quad (23)$$

### WAVE NUMBER OF THE LIGHT ALONG PATH B

Let  $\lambda_0'$  be the wave number of the light along path B. This will also be the wave number of the light along the path BT or B'. The path B' will be discussed in detail in a later section. Applying equation (21) and making the same substitution for  $\frac{1}{\lambda_0}$  as was made in equation (22) we obtain:

$$\frac{1}{\lambda_0'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta_0 + \phi_0) \cos \theta_0 + \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0) - \dots \right]$$

Substituting equation (22) for  $\frac{1}{\lambda_0}$  in this relation:

$$\frac{1}{\lambda_0'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta_0 + \phi_0) \cos \theta_0 + \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0) + \dots \right]$$

Replace  $\lambda_0'$  by equation (23).

$$\frac{1}{\lambda_0'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta_0 + \phi_0) \cos \theta_0 + \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0) + \dots \right]$$

Now substitute equation (22) for  $\frac{1}{\lambda_0}$ . This yields:

$$\frac{1}{\lambda_0'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta_0 + \phi_0) \cos \theta_0 + \frac{v^2}{c^2} \cos^2(\theta_0 + \phi_0) + \dots \right] + \frac{v^2}{c^2} \left[ \frac{1}{2} \cos^2 \theta_0 - \frac{1}{2} \cos^2(\theta_0 + \phi_0) + \sin(\theta_0 + \phi_0) \sin \theta_0 \cos \theta_0 \right]$$



The latter reduces to :

$$\frac{1}{\lambda'_2} = \frac{1}{\lambda_0} \left[ 1 - 2 \frac{v}{c} \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \frac{v^2}{c^2} \cos \varphi_0 \left\{ \frac{1}{2} \cos \varphi_0 + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \right\} + \dots \right]$$

Eqn. (70)

Finally, we must apply the Lorentz transformation to  $1/\lambda'_2$  itself. In figure I, we see that the angle between  $D'_1$  or  $D'_3$  and the direction of the velocity  $(\theta + \theta' + \phi + i'_2 + \sigma)$ . To the order of approximation that it is needed, the cosine of this latter angle is  $\cos(2\theta_0 + \phi_0)$ . Therefore, applying equation 1, we obtain

$$\frac{1}{\lambda'_2} = \frac{1}{\lambda'_{2_0}} \left[ 1 - \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) \right]^{-\frac{1}{2}}$$

Expanding, we get:

$$\frac{1}{\lambda'_2} = \frac{1}{\lambda'_{2_0}} \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) + \dots \right]$$

Substituting the latter expression in equation 70 we obtain as the final equation of the wave number along path  $D'_1$  and  $D'_3$  :

$$\begin{aligned} \frac{1}{\lambda'_2} = \frac{1}{\lambda_0} & \left[ 1 - 2 \frac{v}{c} \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2 \varphi_0 - \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \cos \varphi_0 \right\} + \dots \right] \end{aligned}$$

Eqn. (71)

For later reference, we shall write this equation symbolically as:

$$\frac{1}{\lambda'_{2_0}} = \frac{1}{\lambda_0} \left[ 1 + \frac{v}{c} C + \frac{v^2}{c^2} D + \dots \right]$$

Eqn. (72)

where C and D represent the coefficients of the  $(v/c)^1$  and the  $(v/c)^2$  terms, respectively.

The latter reduces to :

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta + \phi) \cos \theta_0 \right] + \frac{v^2}{c^2} \cos \theta_0 \left[ \frac{1}{2} \cos(\theta + \phi) \cos \theta_0 + \dots \right]$$

Eqn. (70)

Finally, we must apply the Lorentz transformation to  $\lambda_0$ . As seen in figure 1, we see that the angle between  $D_1'$  or  $D_2'$  and the direction of the velocity is  $(\theta + \phi) + \frac{1}{2}(\theta + \phi)$ . To the order of approximation that it is needed, the cosine of this latter angle is  $\cos(\theta_0 + \frac{1}{2}(\theta + \phi))$ . Therefore, applying equation 1, we obtain

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta_0 + \frac{1}{2}(\theta + \phi)) \right] + \frac{v^2}{c^2} \cos \theta_0 \left[ \frac{1}{2} \cos(\theta_0 + \frac{1}{2}(\theta + \phi)) + \dots \right]$$

Repeating, we get:

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0} \left[ 1 + \frac{1}{2} \frac{v}{c} \cos^2(\theta_0 + \phi) + \dots \right]$$

Substituting the latter expression in equation 70 we obtain as the final equation of the wave number along path  $D_1'$  and  $D_2'$  :

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta + \phi) \cos \theta_0 \right] + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2 \theta_0 - \frac{1}{2} \cos(\theta + \phi) \cos \theta_0 \cos \theta_0 + \dots \right\}$$

Eqn. (71)

For later reference, we shall write this equation symbolically as:

$$\frac{1}{\lambda'} = \frac{1}{\lambda_0} \left[ 1 + \frac{v}{c} C + \frac{v^2}{c^2} D + \dots \right]$$

Eqn. (72)

where  $C$  and  $D$  represent the coefficients of the  $(v/c)^1$  and the  $(v/c)^2$  terms, respectively.



### WAVE NUMBER OF LIGHT ALONG PATH D<sub>1</sub>'

Since the light which is incident upon the mirror M'' has not undergone any reflection, its wave number is simply that of the light which the moving observer starts with, namely:

$$\frac{1}{\lambda_0}.$$

### WAVE NUMBER OF LIGHT ALONG PATH D<sub>2</sub>'

To find the wave number of the light along path D<sub>2</sub>', apply equation 23. Let  $1/\lambda_2''$  be the wave number of the light reflected at M'' and  $1/\lambda_0$  the wave number of the incident light. Referring to equation 47, we see that  $v/c \sin \phi$  must be replaced by  $-v/c \cos(\phi - \omega)$ . Therefore:

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda} \left[ 1 - 2 \frac{v}{c} \cos(\phi - \omega) \cos j_1 + 2 \frac{v^2}{c^2} \cos^2(\phi - \omega) + \dots \right]$$

Substitute for  $\cos(\phi - \omega)$  and  $\cos j_1$ , their values given on page 47. Also replace  $1/\lambda$  by its value given in equation 66.

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda_0} \left[ 1 - 2 \frac{v}{c} \cos \phi_0 + \frac{v^2}{c^2} \left\{ \frac{5}{2} \cos^2 \phi_0 - 2 \sin \phi_0 \sin \mu_0 \right\} + \dots \right] \quad \text{Eqn. (73)}$$

Let us now apply the Lorentz transformation to  $1/\lambda_2''$ . From figure I we found that the angle between D<sub>2</sub>' and the direction of the velocity is  $(\phi + j_1' - \omega)$ . To the order of approximation that it is needed, the cosine of this latter angle is found to be  $\cos \phi_0$ . Therefore:

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda_2''} \left[ 1 - \frac{v^2}{c^2} \cos^2 \phi_0 \right]^{-\frac{1}{2}}$$

# WAVE NUMBER OF LIGHT ALONG PATH II

Since the light which is incident upon the mirror  $M$  has not undergone any reflection, its wave number is simply that of the light which the moving observer emits along path I, namely:

$$\frac{1}{\lambda_0}$$

# WAVE NUMBER OF LIGHT ALONG PATH II

To find the wave number of the light along path II, apply equation 10, let  $\lambda_0'$  be the wave number of the light reflected at  $M'$  and  $\lambda_0$  the wave number of the incident light. Referring to equation 17, we see that  $v/c \sin \theta$  must be replaced by  $-v/c \cos(\theta - \omega)$ . Therefore:

$$\frac{1}{\lambda_0'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta - \omega) \right] + \frac{v}{c} \cos \theta \cos \omega$$

Substitute for  $\cos(\theta - \omega)$  and use  $\lambda_0'$  and  $\lambda_0$  their values given on page 41. Also replace  $\lambda_0$  by the value given in equation 10.

$$\frac{1}{\lambda_0'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos \theta \cos \omega + \frac{v}{c} \cos \theta \cos \omega \right] + \frac{v}{c} \cos \theta \cos \omega \quad (22)$$

Let us now apply the Lorentz transformation to  $\lambda_0'$ . From

Figure 1 we found that the angle between  $\vec{M}'$  and the direction of the velocity is  $(\theta' + \theta) - \omega$ . To the order of approximation that is needed, the value of this angle must be taken to be  $\theta$ . Therefore:

$$\frac{1}{\lambda_0'} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos \theta \cos \omega \right] + \frac{v}{c} \cos \theta \cos \omega$$



Expanding, we obtain:

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda_2''} \left[ 1 + \frac{v}{c} \cos^2 \varphi_0 + \dots \right].$$

Replacing  $1/\lambda_1''$  in equation 73 by this latter expression, we obtain as the final equation of the wave number of the light along path  $D_2''$  :

$$\frac{1}{\lambda_2''} = \frac{1}{\lambda_0} \left[ 1 - 2 \frac{v}{c} \cos \varphi_0 + 2 \frac{v^2}{c^2} \{ \cos^2 \varphi_0 - \sin \varphi_0 \sin \mu_0 \} + \dots \right]. \quad \text{Eqn. (74)}$$

#### WAVE NUMBER OF LIGHT ALONG PATH $D_3''$

The path  $D_3''$  will be discussed in detail in a later section. Let  $1/\lambda_3''$  be the wave number of the light along the path  $D_3''$ . Applying equation 23, and making the same substitution for  $v/c \sin \phi$  as was made in equation 53, we obtain:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_2''} \left[ 1 + 2 \frac{v}{c} \sin(\vartheta + \varphi) \cos j_2 + 2 \frac{v^2}{c^2} \sin^2(\vartheta + \varphi) + \dots \right],$$

Replace  $\cos j_2$  by its value given on page 51. Also substitute equation 5 for  $\sin(\vartheta + \phi)$ . This yields:

$$\begin{aligned} \frac{1}{\lambda_3''} = \frac{1}{\lambda_2''} & \left[ 1 + 2 \frac{v}{c} \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \right. \\ & \left. + 2 \frac{v^2}{c^2} \sin(\vartheta_0 + \varphi_0) \{ \sin(\vartheta_0 + \varphi_0) - \cos \vartheta_0 \sin \varphi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} + \dots \right], \end{aligned}$$

Simplifying, this becomes:

$$\begin{aligned} \frac{1}{\lambda_3''} = \frac{1}{\lambda_2''} & \left[ 1 + 2 \frac{v}{c} \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \right. \\ & \left. + 2 \frac{v^2}{c^2} \sin(\vartheta_0 + \varphi_0) \{ \sin \vartheta_0 \cos \varphi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} + \dots \right]. \end{aligned} \quad \text{Eqn. (75)}$$

Expanding, we obtain:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_2} \left[ 1 + \frac{v}{c} \cos \theta_2' + \dots \right]$$

Replacing  $\lambda_2'$  in equation 73 by this latter expression, we obtain as the

final equation of the wave number of the light along path  $P_2'$ :

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_2} \left[ 1 - \frac{v}{c} \cos \theta_2 + \frac{v^2}{c^2} \left\{ \cos^2 \theta_2 - 2 \cos \theta_2 \cos \theta_2' + \cos^2 \theta_2' \right\} + \dots \right] \quad \text{Eqn. (74)}$$

### WAVE NUMBER OF LIGHT ALONG PATH $P_2'$

The path  $P_2'$  will be discussed in detail in a later section. Let

$\lambda_2'$  be the wave number of the light along the path  $P_2'$ . Applying equation

73, and making the same substitution for  $v$  as was made in equation

63, we obtain:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_2} \left[ 1 + \frac{v}{c} \sin \theta_2 \cos \theta_2' + \frac{v^2}{c^2} \sin^2 \theta_2 \cos^2 \theta_2' + \dots \right]$$

Replace now  $\lambda_2'$  by its value given in eqn. 74. Also substitute equation 7 for

$\sin(\theta_2 + \theta_2')$ . This yields:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_2} \left[ 1 + \frac{v}{c} \sin \theta_2 \cos \theta_2' + \frac{v^2}{c^2} \left\{ \sin^2 \theta_2 \cos^2 \theta_2' - \cos^2 \theta_2 \cos^2 \theta_2' + 2 \cos \theta_2 \cos \theta_2' \sin \theta_2 \cos \theta_2' \right\} + \dots \right]$$

Simplifying, this becomes:

$$\frac{1}{\lambda_2'} = \frac{1}{\lambda_2} \left[ 1 + \frac{v}{c} \sin \theta_2 \cos \theta_2' + \frac{v^2}{c^2} \left\{ \sin^2 \theta_2 \cos^2 \theta_2' + \cos^2 \theta_2 \sin^2 \theta_2' + 2 \sin \theta_2 \cos \theta_2' \sin \theta_2 \cos \theta_2' \right\} + \dots \right] \quad \text{Eqn. (75)}$$



Replacing  $1/\lambda_2''$  by its value given in equation 73, yields:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_0} \left[ 1 + 2 \frac{v}{c} \{ \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 - \cos \varphi_0 \} \right. \\ \left. + \frac{v^2}{c^2} \left\{ \frac{5}{2} \cos^2 \varphi_0 - 2 \sin \varphi_0 \sin \mu_0 - 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \varphi_0 \right. \right. \\ \left. \left. + 2 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \varphi_0 + 4 \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \sin \mu_0 \right\} + \dots \right].$$

The latter reduces to :

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_0} \left[ 1 - 2 \frac{v}{c} \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \right. \\ \left. + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2 \varphi_0 + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \cos \varphi_0 + 2 \sin(2\vartheta_0 + \varphi_0) \sin \mu_0 \right\} + \dots \right] \quad \text{Eqn. (76)}$$

Finally, we must apply the Lorentz transformation to  $1/\lambda_3''$ .

Referring to figure I we see that the angle between  $D_3''$  and the direction of the velocity is  $(j_2' + \frac{\pi}{2} - \theta - \phi)$ . To the order of approximation that it is needed, the cosine of this latter angle is found to be  $\cos(2\theta_0 + \phi_0)$ .

Therefore, substituting in equation 1, we get:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_{3_0}''} \left[ 1 - \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) \right]^{-\frac{1}{2}},$$

which becomes upon expansion:

$$\frac{1}{\lambda_3''} = \frac{1}{\lambda_{3_0}''} \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2(2\vartheta_0 + \varphi_0) + \dots \right].$$

Replace  $1/\lambda_3''$  in equation 76 by this latter expression.

$$\frac{1}{\lambda_{3_0}''} = \frac{1}{\lambda_0} \left[ 1 - 2 \frac{v}{c} \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \right. \\ \left. + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2 \varphi_0 - \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) \right. \right. \\ \left. \left. + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \cos \varphi_0 \right. \right. \\ \left. \left. + 2 \sin(2\vartheta_0 + \varphi_0) \sin \mu_0 \right\} + \dots \right]. \quad \text{Eqn. (77)}$$

For later reference, we shall write this equation symbolically as:

$$\frac{1}{\lambda_{3_0}''} = \frac{1}{\lambda_0} \left[ 1 + \frac{v}{c} A + \frac{v^2}{c^2} B + \dots \right],$$

Eqn. (78)

Replacing  $\lambda''$  by its value given in equation 75, yields:

$$\frac{1}{\lambda''} = \frac{1}{\lambda_0} \left[ 1 + \frac{v}{c} \left( \sin \theta_0 \sin \theta_0' - \cos \theta_0 \right) \right] + \frac{v^2}{2c^2} \left\{ \cos^2 \theta_0' - \frac{v}{c} \sin \theta_0 \sin \theta_0' - \frac{1}{2} \sin^2 \theta_0 \sin^2 \theta_0' \cos \theta_0 \right. \\ \left. + 2 \sin \theta_0 \sin \theta_0' \cos \theta_0 + \frac{1}{2} \sin^2 \theta_0 \cos^2 \theta_0' \right\} + \dots$$

The latter reduces to :

$$\frac{1}{\lambda''} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta_0 + \theta_0') \cos \theta_0 \right] + \frac{v^2}{2c^2} \left[ \frac{1}{2} \cos^2 \theta_0' + 2 \cos \theta_0 \cos \theta_0' \cos \theta_0 + \frac{1}{2} \sin^2 \theta_0 \sin^2 \theta_0' \cos \theta_0 \right] + \dots \quad \text{Eqn. (76)}$$

Finally, we must apply the Lorentz transformation to  $\lambda''$ . Referring to Figure 1 we see that the angle between  $\vec{U}$  and the direction of the velocity is  $(\frac{1}{2} - \frac{\pi}{2} - \theta_0)$ . To the order of approximation that it is needed, the cosine of this latter angle is found to be  $\cos(2\theta_0 + \frac{1}{2}\pi)$ . Therefore, substituting in equation 1, we get:

$$\frac{1}{\lambda''} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos^2(\theta_0 + \theta_0') \right] - \frac{1}{2}$$

which becomes upon expansion:

$$\frac{1}{\lambda''} = \frac{1}{\lambda_0} \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2(\theta_0 + \theta_0') + \dots \right]$$

Replace  $\lambda''$  in equation 76 by this latter expression.

$$\frac{1}{\lambda''} = \frac{1}{\lambda_0} \left[ 1 - \frac{v}{c} \cos(\theta_0 + \theta_0') \cos \theta_0 \right] + \frac{v^2}{2c^2} \left\{ \frac{1}{2} \cos^2 \theta_0' - \frac{1}{2} \cos^2 \theta_0 \sin^2 \theta_0' \cos \theta_0 \right. \\ \left. + 2 \cos \theta_0 \cos \theta_0' \cos \theta_0 + \frac{1}{2} \sin^2 \theta_0 \sin^2 \theta_0' \cos \theta_0 \right\} + \dots \quad \text{Eqn. (77)}$$

For later reference, we shall write this equation symbolically as:

$$\frac{1}{\lambda''} = \frac{1}{\lambda_0} \left[ 1 + \frac{v}{c} A + \frac{v^2}{2c^2} B + \dots \right]$$

Eqn. (78)



where A and B represent the coefficient of the  $(v/c)^1$  and the  $(v/c)^2$  terms respectively.

Comparing equations 71 and 77 we see that except for the term involving  $\mu_0$ , the two final wave numbers are equal to the second order of  $v/c$ .

#### NUMBER OF WAVES IN THE PATHS $D_1'$ , $D_2'$ , $D_1''$ AND $D_2''$

In order to compare the number of waves in paths AC + CH - HT and AP + PL - LT it is necessary to multiply each part of the path by the wave number of the light contained within that path.

Let  $N_1'$  be defined as the number of wave lengths in the path  $D_1'$ .

Then:

$$N_1' = \frac{D_1'}{\lambda_1'}$$

To find  $N_1'$  multiply equation 61 by equation 69.

$$N_1' = \frac{s_0'}{\lambda_0} \left[ 1 + \frac{v}{c} \{ \cos(2\theta_0 + \phi_0) - 2 \sin(\theta_0 + \phi_0) \sin \theta_0 \} \right. \\ \left. + \frac{v^2}{c^2} \left\{ \frac{1}{2} - 2 \cos(2\theta_0 + \phi_0) \sin(\theta_0 + \phi_0) \sin \theta_0 + \frac{1}{2} \cos^2 \phi_0 \right. \right. \\ \left. \left. - \frac{1}{2} \cos^2(2\theta_0 + \phi_0) + 2 \{ \sin(\theta_0 + \phi_0) \sin \theta_0 \cos \phi_0 \} \dots \right\} \right].$$

Upon simplification, this becomes:

$$N_1' = \frac{s_0'}{\lambda_0} \left[ 1 + \frac{v}{c} \{ \cos(2\theta_0 + \phi_0) - 2 \sin(\theta_0 + \phi_0) \sin \theta_0 \} \right. \\ \left. + \frac{v^2}{c^2} \left\{ \frac{1}{2} \sin^2(2\theta_0 + \phi_0) + \frac{1}{2} \cos^2 \phi_0 \right. \right. \\ \left. \left. + 2 \sin(\theta_0 + \phi_0) \sin \theta_0 [\cos \phi_0 - \cos(2\theta_0 + \phi_0)] \right\} + \dots \right].$$

Eqn. (79)

Let  $N_2'$  be defined as the number of wave lengths in the path  $D_2'$

Then:

$$N_2' = \frac{D_2'}{\lambda_2'}$$



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To find  $N_2'$  multiply equation 62 by equation 71.

$$N_2' = \frac{S_0'}{\lambda_0} \left[ 1 - \frac{v}{c} \{ \cos(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \right. \\ \left. + \frac{v^2}{c^2} \left\{ \frac{1}{2} + 2 \cos(2\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \frac{1}{2} \cos^2 \varphi_0 \right. \right. \\ \left. \left. + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \cos \varphi_0 - \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) \right\} + \dots \right]$$

This reduces to :

$$N_2' = \frac{S_0'}{\lambda_0} \left[ 1 - \frac{v}{c} \{ \cos(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \right. \\ \left. + \frac{v^2}{c^2} \left\{ \frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) + \frac{1}{2} \cos^2 \varphi_0 \right. \right. \\ \left. \left. + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 [\cos \varphi_0 + \cos(2\vartheta_0 + \varphi_0)] \right\} + \dots \right] \\ \text{Eqn. (80)}$$

Let  $N_1'$  be defined as the number of wave lengths in path  $D_1'$  .

Then:

$$N_1' = \frac{D_1'}{\lambda_0}$$

To find  $N_1''$  multiply equation 63 by  $1/\lambda_0$ .

$$N_1'' = \frac{S_0''}{\lambda_0} \left[ 1 + \frac{v}{c} \cos \varphi_0 + \frac{1}{2} \frac{v^2}{c^2} + \dots \right]. \\ \text{Eqn. (81)}$$

Let  $N_2''$  be defined as the number of wave lengths in path  $D_2''$  .

Then:

$$N_2'' = \frac{D_2''}{\lambda_2''}$$

To find  $N_2''$  multiply equation 64 by equation 74.

$$N_2'' = \frac{S_0''}{\lambda_0} \left[ 1 - \frac{v}{c} \frac{1}{\sin \vartheta_0} \{ \sin \vartheta_0 \cos \varphi_0 + 2 \cos \vartheta_0 \sin \mu_0 + 2 \sin \vartheta_0 \cos \varphi_0 \} \right. \\ \left. + \frac{v^2}{c^2} \frac{1}{\sin^2 \vartheta_0} \left\{ \frac{1}{2} \sin^2 \vartheta_0 + 2 \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \sin \mu_0 \right. \right. \\ \left. \left. + 2(1 + \cos^2 \vartheta_0) \sin^2 \mu_0 + 2 \sin^2 \vartheta_0 \cos^2 \varphi_0 \right. \right. \\ \left. \left. + 4 \sin \vartheta_0 \cos \vartheta_0 \cos \varphi_0 \sin \mu_0 + 2 \sin^2 \vartheta_0 \cos^2 \varphi_0 \right. \right. \\ \left. \left. - 2 \sin^2 \vartheta_0 \sin \varphi_0 \sin \mu_0 \right\} + \dots \right]$$

Upon simplification, this reduces to :

To find  $N_1'$  multiply equation 61 by  $\sin \theta_0$  and

$$N_1' = \frac{\partial}{\partial \theta_0} \left[ 1 - \frac{\gamma}{2} \left\{ \cos(\theta_0 + \theta_0') + \cos(\theta_0 - \theta_0') \right\} \right] \\ + \frac{\gamma}{2} \left[ \frac{1}{2} + \cos(\theta_0 + \theta_0') \cos \theta_0' + \frac{1}{2} \cos \theta_0' \right] \\ + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' - \frac{1}{2} \cos(\theta_0 - \theta_0') \cos \theta_0' ]$$

and reduce to :

$$N_1' = \frac{\partial}{\partial \theta_0} \left[ 1 - \frac{\gamma}{2} \left\{ \cos(\theta_0 + \theta_0') + \cos(\theta_0 - \theta_0') \right\} \right] \\ + \frac{\gamma}{2} \left[ \frac{1}{2} \sin \theta_0' + \frac{1}{2} \cos \theta_0' \right] \\ + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' - \frac{1}{2} \cos(\theta_0 - \theta_0') \cos \theta_0' ]$$

Eqn. (60)

Int  $N_1'$  w.r.t  $\theta_0$  on the number 1 and integrate it upto  $\theta_0'$ .

Then:

$$N_1'' = \frac{\partial N_1'}{\partial \theta_0}$$

To find  $N_1''$  multiply equation 60 by  $\sin \theta_0$

$$N_1'' = \frac{\partial}{\partial \theta_0} \left[ 1 + \frac{\gamma}{2} \cos \theta_0' + \frac{1}{2} \cos \theta_0' \right]$$

Eqn. (61)

Int  $N_1''$  w.r.t  $\theta_0$  on the number 1 and integrate it upto  $\theta_0'$ .

Then:

$$N_1''' = \frac{\partial N_1''}{\partial \theta_0}$$

To find  $N_1'''$  multiply equation 61 by  $\sin \theta_0$  and

$$N_1''' = \frac{\partial}{\partial \theta_0} \left[ 1 - \frac{\gamma}{2} \left\{ \frac{1}{2} \sin \theta_0' + \frac{1}{2} \cos \theta_0' \right\} + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' \right. \\ \left. + \frac{\gamma}{2} \left[ \frac{1}{2} \sin \theta_0' + \frac{1}{2} \cos \theta_0' \right] + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' \right. \\ \left. + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' \right. \\ \left. + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' + \gamma \cos(\theta_0 + \theta_0') \cos \theta_0' \right]$$

and differentiate, the result is :



$$N_2'' = \frac{S_0''}{\lambda_0} \left[ 1 - \frac{v}{c} \frac{1}{\sin \vartheta_0} \{ 3 \sin \vartheta_0 \cos \varphi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} \right. \\ \left. + \frac{v^2}{c^2} \frac{1}{\sin^2 \vartheta_0} \left\{ \frac{1}{2} \sin^2 \vartheta_0 (1 + 8 \cos^2 \varphi_0) + 2 \sin \vartheta_0 \sin \mu_0 (2 \cos(\vartheta_0 + \varphi_0) + \cos \vartheta_0 \cos \varphi_0) \right. \right. \\ \left. \left. + 2(1 + \cos^2 \vartheta_0) \sin^2 \mu_0 \right\} + \dots \right] \quad \text{Eqn. (82)}$$

Having obtained the number of wave lengths along the paths  $D_1'$ ,  $D_2'$ ,  $D_1''$  and  $D_2''$ ; in terms of the wave length of the light with which the moving observer is working, we may now find the difference between the number of wave lengths in corresponding parts of the two paths. Let:

$$N_1'' - N_1' = N_1$$

Subtracting equation 79 from equation 81, and multiplying both sides by  $\lambda_0$ , we get:

$$N_1 \lambda_0 = (S_0'' - S_0') + \frac{v}{c} [S_0'' \cos \varphi_0 - S_0' \{ \cos(2\vartheta_0 + \varphi_0) - 2 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \}] \\ + \frac{v^2}{c^2} \left[ \frac{1}{2} S_0'' - S_0' \left\{ \frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) + \frac{1}{2} \cos^2 \varphi_0 \right. \right. \\ \left. \left. + 2 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 [\cos \varphi_0 - \cos(2\vartheta_0 + \varphi_0)] \right\} \right] + \dots \quad \text{Eqn. (83)}$$

Similarly, let :

$$N_2'' - N_2' = N_2$$

Subtracting equation 80 from equation 82, and multiplying both sides by  $\lambda_0$ , we get:

$$N_2 \lambda_0 = (S_0'' - S_0') - \frac{v}{c} \left[ S_0'' \frac{1}{\sin \vartheta_0} \{ 3 \sin \vartheta_0 \cos \varphi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} \right. \\ \left. - S_0' \{ \cos(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \right] \\ + \frac{v^2}{c^2} \left[ S_0'' \frac{1}{\sin^2 \vartheta_0} \left\{ \frac{1}{2} \sin^2 \vartheta_0 (1 + 8 \cos^2 \varphi_0) + 2 \sin \vartheta_0 \sin \mu_0 (2 \cos(\vartheta_0 + \varphi_0) + \cos \vartheta_0 \cos \varphi_0) \right. \right. \\ \left. \left. + 2(1 + \cos^2 \vartheta_0) \sin^2 \mu_0 \right\} \right. \\ \left. - S_0' \left\{ \frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) + \frac{1}{2} \cos^2 \varphi_0 \right. \right. \\ \left. \left. + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 [\cos \varphi_0 + \cos(2\vartheta_0 + \varphi_0)] \right\} \right] + \dots \quad \text{Eqn. (84)}$$

$$N_1'' = \frac{2}{\lambda} \left( 1 - \frac{1}{2} \frac{1}{\sin^2 \theta} \left[ 2 \sin^2 \theta \cos^2 \theta + 2 \cos^2 \theta \sin^2 \theta \right] \right)$$

$$+ \frac{1}{2} \frac{1}{\sin^2 \theta} \left[ \frac{1}{2} \sin^2 \theta (1 + \cos^2 \theta) + 2 \sin^2 \theta \cos^2 \theta + 2 \cos^2 \theta \sin^2 \theta \right]$$

$$+ 2(1 + \cos^2 \theta) \sin^2 \theta + \dots \quad \text{Eqn. (28)}$$

Being obtained the number of wave lengths along the paths  $O_1$  &  $O_2$  and  $O_1'$  &  $O_2'$ ; in terms of the wave length of the light with which the working observer is working, we may now find the difference between the number of wave lengths in corresponding parts of the two paths. Let:

$$N_1'' - N_1' = N_1$$

Substituting equation 27 from equation 26, and multiplying both

sides by  $\lambda$ , we get:

$$N_1 \lambda = (2'' - 2') + \frac{1}{2} [2'' (1 + \cos^2 \theta) - 2' (1 + \cos^2 \theta) + 2 \sin^2 \theta \cos^2 \theta]$$

$$+ \frac{1}{2} [2'' (1 + \cos^2 \theta) - 2' (1 + \cos^2 \theta) + 2 \sin^2 \theta \cos^2 \theta]$$

$$+ 2 \sin^2 \theta \cos^2 \theta (1 + \cos^2 \theta) + \dots$$

Eqn. (29)

Similarly, let:

$$N_2'' - N_2' = N_2$$

Substituting the equation for the equation 26, and multiplying both sides by  $\lambda$ , we get:

$$N_2 \lambda = (2'' - 2') + \frac{1}{2} [2'' (1 + \cos^2 \theta) - 2' (1 + \cos^2 \theta) + 2 \sin^2 \theta \cos^2 \theta]$$

$$- 2' [2'' (1 + \cos^2 \theta) - 2' (1 + \cos^2 \theta) + 2 \sin^2 \theta \cos^2 \theta]$$

$$+ \frac{1}{2} [2'' (1 + \cos^2 \theta) - 2' (1 + \cos^2 \theta) + 2 \sin^2 \theta \cos^2 \theta]$$

$$+ 2 \sin^2 \theta \cos^2 \theta (1 + \cos^2 \theta) + \dots$$

$$- 2' [2'' (1 + \cos^2 \theta) - 2' (1 + \cos^2 \theta) + 2 \sin^2 \theta \cos^2 \theta]$$

$$+ 2 \cos^2 \theta (1 + \cos^2 \theta) [2'' (1 + \cos^2 \theta) + \dots]$$

Eqn. (30)



Let us now add equations 83 and 84 to get the total difference in path length along the first two parts of each path.

$$\begin{aligned}(N_1 + N_2)\lambda_0 = & 2(s_0'' - s_0') + \frac{v}{c} \left[ -s_0'' \frac{2}{\sin \vartheta_0} \{ \sin \vartheta_0 \cos \varphi_0 + \cos \vartheta_0 \sin \mu_0 \} - 2s_0' \cos \varphi_0 \right] \\ & + \frac{v^2}{c^2} \left[ s_0'' \frac{1}{\sin^2 \vartheta_0} \{ \sin^2 \vartheta_0 (1 + 4 \cos^2 \varphi_0) \right. \\ & \quad \left. + 2 \sin \vartheta_0 \sin \mu_0 [2 \cos(\vartheta_0 + \varphi_0) + \cos \vartheta_0 \cos \varphi_0] \right. \\ & \quad \left. + 2(1 + \cos^2 \vartheta_0) \sin^2 \mu_0 \right. \\ & \quad \left. - s_0' \{ \sin^2(2\vartheta_0 + \varphi_0) + 3 \cos^2 \varphi_0 + 2 \cos^2(2\vartheta_0 + \varphi_0) \} \right] + \dots\end{aligned}$$

Simplifying, we obtain:

$$\begin{aligned}(N_1 + N_2)\lambda_0 = & 2(s_0'' - s_0') + \frac{v}{c} \left[ -2(s_0'' - s_0') \cos \varphi_0 - 2s_0'' \frac{1}{\sin \vartheta_0} \cos \vartheta_0 \sin \mu_0 \right] \\ & + \frac{v^2}{c^2} \left[ (s_0'' - s_0') (1 + 3 \cos^2 \varphi_0) \right. \\ & \quad \left. + s_0'' \frac{2}{\sin^2 \vartheta_0} \{ \sin \vartheta_0 \sin \mu_0 [2 \cos(\vartheta_0 + \varphi_0) + \cos \vartheta_0 \cos \varphi_0] \right. \\ & \quad \left. + \sin^2 \vartheta_0 \cos^2 \varphi_0 + (1 + \cos^2 \vartheta_0) \sin^2 \mu_0 \right. \\ & \quad \left. - s_0' \cos^2(2\vartheta_0 + \varphi_0) \right] + \dots\end{aligned}\quad \text{Eqn. (85)}$$

#### DIFFERENCE BETWEEN NUMBER OF WAVE LENGTHS IN THE TWO FINAL PATHS

In examining the equation for  $\sin \epsilon$  (Eqn. 58), we find that to the first power of  $v/c$  this angle is a function of  $\mu_0$ , the angle introduced at the mirror  $M''$ . The second order term of  $\sin \epsilon$ , we find, is due partially to  $\mu_0$  and partially to the angles  $\vartheta_0$  and  $\varphi_0$ . Because of this small angle  $\epsilon$  the two final rays seem to come from a point T in figure I. Hence, in order to evaluate the number of wave lengths in the path designated by the single primes, it is necessary to subtract from  $N_1'$  the number of wave lengths contained within the distance HT. Let:

$$HT = D_3'$$

Let us now add equations (2) and (3) to get the total distance in

path length along the line in units of mean path.

$$(M+1)\lambda_e = 2(2\pi - 2\pi) + \frac{2}{\pi} \left[ -2\pi \cos \phi_e + \frac{2}{\pi} \left( \sin \phi_e \cos \phi_e + \cos \phi_e \sin \phi_e \right) - 2\pi \cos \phi_e \right]$$

$$+ \frac{2}{\pi} \left[ 2\pi \cos \phi_e + \frac{2}{\pi} \left( \sin \phi_e \cos \phi_e + \cos \phi_e \sin \phi_e \right) - 2\pi \cos \phi_e \right]$$

$$+ 2\pi \sin \phi_e \cos \phi_e + 2\pi \cos \phi_e \sin \phi_e$$

$$+ 2\pi \sin \phi_e \cos \phi_e + 2\pi \cos \phi_e \sin \phi_e$$

$$- 2\pi \left[ \sin \phi_e \cos \phi_e + \cos \phi_e \sin \phi_e + 2\pi \cos \phi_e + 2\pi \cos \phi_e \right]$$

Thus we obtain:

$$(M+1)\lambda_e = 2(2\pi - 2\pi) + \frac{2}{\pi} \left[ -2\pi \cos \phi_e + \frac{2}{\pi} \left( \sin \phi_e \cos \phi_e + \cos \phi_e \sin \phi_e \right) - 2\pi \cos \phi_e \right]$$

$$+ \frac{2}{\pi} \left[ 2\pi \cos \phi_e + \frac{2}{\pi} \left( \sin \phi_e \cos \phi_e + \cos \phi_e \sin \phi_e \right) - 2\pi \cos \phi_e \right]$$

$$+ \frac{2}{\pi} \left[ 2\pi \sin \phi_e \cos \phi_e + 2\pi \cos \phi_e \sin \phi_e - 2\pi \cos \phi_e \right]$$

$$+ 2\pi \sin \phi_e \cos \phi_e + 2\pi \cos \phi_e \sin \phi_e$$

$$- 2\pi \cos \phi_e \sin \phi_e - 2\pi \sin \phi_e \cos \phi_e$$

### RELATIONSHIP BETWEEN NUMBER OF WAVE LENGTHS IN THE TWO WAVE PATHS

In examining the equation for wave (2) in (3), we find that in the first order of approximation the angle is a function of  $M$ , the angle introduced in the mirror  $M$ . The second order term of this  $\phi$ , we find, is due partially to  $M$  and partially to the angle  $\phi_e$ . Because of this small angle  $\phi_e$  the two wave rays seem to come from a point  $T$  in figure 2. Hence, in order to evaluate the number of wave lengths in the path described by the mirror system, it is necessary to subtract from  $M$  the number of wave lengths contained within the distance  $MT$ . Let:

$$MT = D_1$$



Similarly, to evaluate the number of wave length in the path designated by the double primes, it is necessary to subtract from  $N_2''$  the number of wave lengths contained within the distance  $LT$ . Let:

$$LT = D_3'',$$

We shall now evaluate the distances  $D_3'$  and  $D_3''$ . In triangle  $TLG$  of figure I, we have by the sine law:

$$\frac{\sin \epsilon}{LG} = \frac{\sin(\frac{\pi}{2} - i_3)}{D_3''} = \frac{\sin(\frac{\pi}{2} + j_1)}{D_3' + HG}.$$

Eqn. (86)

Solving for  $D_3''$ , we get:

$$D_3'' = LG \frac{\cos i_3}{\sin \epsilon}$$

Replacing  $i_3$  by its value  $(\frac{\pi}{2} - \theta' - \sigma - i_1')$ , we get:

$$D_3'' = LG \frac{\sin(\theta' + \sigma + i_1')}{\sin \epsilon}$$

Substitute for  $\sin \epsilon$  its equal  $v/c \sin \epsilon'$ . This yields:

$$D_3'' = \frac{LG}{c} \frac{\sin(\theta' + \sigma + i_1')}{\sin \epsilon'}$$

The wave number of the light along the path  $D_3''$  was represented in equation 78 as :

$$\frac{1}{\lambda_{3_0}''} = \frac{1}{\lambda_0} \left( 1 + \frac{v}{c} A + \frac{v^2}{c^2} B + \dots \right)$$

Let  $N_3''$  be defined as the number of waves in the path  $D_3''$ , Then:

$$N_3'' = \frac{D_3''}{\lambda_{3_0}''}$$

Similarly, to evaluate the number of wave lengths in the path designated by the double process, it is necessary to subtract from  $N_1$  the number of wave lengths contained within the distance  $LT$ . Let:

$$LT = D_1''$$

We shall now evaluate the distance  $D_1''$  and  $D_2''$ . In triangle  $T_1D_1''$

of figure 1, we have by the sine law:

$$\frac{\sin \epsilon}{D_1''} = \frac{\sin(\frac{\pi}{2} - \epsilon)}{D_1 + H\epsilon} = \frac{\sin(\frac{\pi}{2} + \epsilon)}{D_1 + H\epsilon}$$

Eq. (55)

Solving for  $D_1''$ , we get:

$$D_1'' = L\epsilon \frac{\cos \epsilon}{\sin \epsilon}$$

Replacing  $\epsilon$  by its value  $(\frac{\pi}{2} - \epsilon' - \alpha + \frac{1}{2})$ , we get:

$$D_1'' = L\epsilon \frac{\sin(\frac{1}{2} + \alpha - \epsilon')}{\sin \epsilon}$$

Distance for sine the same  $\epsilon'$ . This yields:

$$D_2'' = \frac{L\epsilon}{\epsilon} \frac{\sin(\frac{1}{2} + \alpha - \epsilon')}{\sin \epsilon'}$$

The wave number of the light along the path  $D_1''$  was represented in equation

as:

$$\frac{1}{\lambda_1''} = \frac{1}{\lambda_0} \left( 1 + \frac{\alpha}{\epsilon} A + \frac{\alpha}{\epsilon} B + \dots \right)$$

Let  $N_2''$  be defined as the number of waves in the path  $D_2''$ . Then:

$$N_2'' = \frac{D_2''}{\lambda_2''}$$



Therefore:

$$N_3'' = \frac{LG}{\frac{v}{c} \lambda_0} \left(1 + \frac{v}{c} A + \frac{v^2}{c^2} B + \dots\right) \frac{\sin(\theta' + \sigma + i_2')}{\sin \epsilon'}$$

Now, substitute equations 44 and 60 for  $\sin(\theta' + \sigma + i_2')$  and  $1/\sin \epsilon'$ .

$$N_3'' = \frac{LG}{\frac{v}{c} \lambda_0} \left(1 + \frac{v}{c} A + \frac{v^2}{c^2} B\right) \left(J + \frac{v}{c} K + \frac{v^2}{c^2} M\right) \left(E + \frac{v}{c} F + \frac{v^2}{c^2} P\right).$$

Multiplying, we obtain:

$$N_3'' = \frac{LG}{\frac{v}{c} \lambda_0} \left[ EJ + \frac{v}{c} (EK + FJ + EAJ) + \frac{v^2}{c^2} (EM + KF + JP + EAK + BJE + FAJ) + \dots \right].$$

Eqn. (87)

Similarly, from equation 86, we obtain:

$$D_3' = LG \frac{\cos j_1'}{\sin \epsilon} - HG.$$

Substitute for  $\sin \epsilon$  its value  $v/c \sin \epsilon'$  Then:

$$D_3' = \frac{LG}{\frac{v}{c}} \frac{\cos j_1'}{\sin \epsilon'} - HG.$$

The wave number of the light along the path  $D_3'$  was represented in equation 72, as:

$$\frac{1}{\lambda_{20}'} = \frac{1}{\lambda_0} \left[1 + \frac{v}{c} C + \frac{v^2}{c^2} D + \dots\right].$$

Let  $N_3'$  be defined as the number of waves in the path  $D_3'$ . Then:

$$N_3' = \frac{D_3'}{\lambda_{20}'}.$$

Therefore:

$$N_2'' = \frac{1}{2} \frac{L}{\lambda} \left( 1 + \frac{v}{c} A + \frac{v}{c} B + \dots \right) \frac{\sin(\delta' + \nu + \epsilon)}{\sin \epsilon}$$

Now, substituting equations 44 and 45 for  $\sin(\delta' + \nu + \epsilon)$  and  $\frac{1}{\sin \epsilon}$ :

$$N_2'' = \frac{1}{2} \frac{L}{\lambda} \left( 1 + \frac{v}{c} A + \frac{v}{c} B \right) \left( 1 + \frac{v}{c} K + \frac{v}{c} M \right) \left( \epsilon + \frac{v}{c} F + \frac{v}{c} G \right)$$

Substituting, we obtain:

$$N_2'' = \frac{1}{2} \frac{L}{\lambda} \left[ \epsilon \left( 1 + \frac{v}{c} (K + F + EA) \right) + \frac{v}{c} (EM + KF + JP + EA + GJ + FAJ) \right]$$

Eq. (47)

Similarly, from equation 46, we obtain:

$$D_1' = L \epsilon \frac{\cos \eta}{\sin \epsilon} - H \epsilon$$

Substituting for  $\sin \epsilon$  the value of  $\sin \epsilon$  from:

$$D_1' = \frac{1}{2} \frac{L}{\lambda} \frac{\cos \eta}{\sin \epsilon} - H \epsilon$$

The value of  $\sin \epsilon$  from the value of  $\sin \epsilon$  can be substituted in equation

47, as:

$$\frac{1}{\lambda} = \frac{1}{\lambda} \left( 1 + \frac{v}{c} C + \frac{v}{c} D + \dots \right)$$

Let  $N_1'$  be defined as the number of waves in the path  $N_1'$ . Then:

$$N_1' = \frac{D_1'}{\lambda}$$



Therefore:

$$N_3' = \frac{LG}{\frac{v}{c}\lambda_0} \left(1 + \frac{v}{c}C + \frac{v^2}{c^2}D + \dots\right) \frac{\cos j_2'}{\sin \epsilon'} - \frac{HG}{\lambda_0} \left(1 + \frac{v}{c}C + \frac{v^2}{c^2}D + \dots\right).$$

Substitute equations 57 and 58 for  $\cos j_2'$  and  $\sin \epsilon'$ .

$$N_3' = \frac{LG}{\frac{v}{c}\lambda_0} \left(1 + \frac{v}{c}C + \frac{v^2}{c^2}D\right) \left(J + \frac{v}{c}K + \frac{v^2}{c^2}M\right) \left(R + \frac{v}{c}S + \frac{v^2}{c^2}T\right) - \frac{HG}{\lambda_0} \left(1 + \frac{v}{c}C + \frac{v^2}{c^2}D\right).$$

Multiplying, we obtain:

$$N_3' = \frac{LG}{\frac{v}{c}\lambda_0} \left[ JR + \frac{v}{c}(RK + JS + RJC) + \frac{v^2}{c^2}(RM + SK + JT + RCK + RJD + SJC) + \dots \right] - \frac{HG}{\lambda_0} \left(1 + \frac{v}{c}C + \frac{v^2}{c^2}D + \dots\right). \quad \text{Eqn. (88)}$$

Let us now subtract the number of wave lengths in the path  $D_3'$  from the number in the path  $D_3''$ . Define  $N_3$ , as:

$$N_3 = N_3'' - N_3'$$

Then:

$$N_3 = \frac{LG}{\frac{v}{c}\lambda_0} \left[ J(E-R) + \frac{v}{c} \{ K(E-R) + J(F-S) + J(EA-RC) \} + \frac{v^2}{c^2} \{ M(E-R) + K(F-S) + J(P-T) + K(EA-RC) + J(BE-RD) + J(FA-SC) \} \right] + \frac{HG}{\lambda_0} \left(1 + \frac{v}{c}C + \frac{v^2}{c^2}D\right) + \dots \quad \text{Eqn. (89)}$$

It will be convenient at this point to summarize the values of the symbols used in the above equation.

$$A = -2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0$$

$$B = \frac{1}{2} \cos^2 \varphi_0 - \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \cos \varphi_0 + 2 \sin(2\vartheta_0 + \varphi_0) \sin \mu_0$$

$$N_2' = \frac{L_2}{\frac{v}{c} \gamma_0} \left( 1 + \frac{v}{c} C + \frac{v}{c} D + \dots \right) - \frac{H_2}{\gamma_0} \left( 1 + \frac{v}{c} C + \frac{v}{c} D + \dots \right)$$

Substituting equations 22 and 23 for  $N_2'$  and  $H_2'$  in eq. 21

$$N_2' = \frac{L_2}{\frac{v}{c} \gamma_0} \left( 1 + \frac{v}{c} C + \frac{v}{c} D \right) \left( 1 + \frac{v}{c} K + \frac{v}{c} L \right) \left( 1 + \frac{v}{c} S + \frac{v}{c} T \right)$$

$$- \frac{H_2}{\gamma_0} \left( 1 + \frac{v}{c} C + \frac{v}{c} D \right)$$

Therefore, we obtain:

$$N_2' = \frac{L_2}{\frac{v}{c} \gamma_0} [ 1 + \frac{v}{c} (K + S + T + L) + \frac{v}{c} (KL + KS + LT + SL) + \frac{v}{c} (KLM + 2KLT + KSL + 2SLT) + \dots ] - \frac{H_2}{\gamma_0} \left( 1 + \frac{v}{c} C + \frac{v}{c} D + \dots \right)$$

Eq. (24)

Let us now subtract the number of wave lengths in the path  $N_2'$

from the number in the path  $N_2''$ . Let  $N_2'''$  be:

$$N_2''' = N_2'' - N_2'$$

Thus:

$$N_2''' = \frac{L_2}{\frac{v}{c} \gamma_0} \left\{ \frac{v}{c} (K + S + T + L) + \frac{v}{c} (KL + KS + LT + SL) + \frac{v}{c} (KLM + 2KLT + KSL + 2SLT) + \dots \right\} - \frac{H_2}{\gamma_0} \left( 1 + \frac{v}{c} C + \frac{v}{c} D + \dots \right)$$

Eq. (25)

It will be convenient at this point to summarize the values of the

variables used in the above equation.

$$K = -1 \cos(\theta_1 + \theta_2) \cos \theta_0$$

$$S = \frac{1}{2} \cos \theta_0 - \frac{1}{2} \cos(\theta_1 + \theta_2) + \frac{1}{2} \cos \theta_1 + \frac{1}{2} \cos \theta_2$$

$$+ \frac{1}{2} \sin \theta_1 \sin \theta_2$$



$$C = -2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0$$

$$D = \frac{1}{2} \cos^2 \varphi_0 - \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \cos \varphi_0$$

$$E = \sin \vartheta_0$$

$$F = \sin(2\vartheta_0 + \varphi_0) \cos \vartheta_0$$

$$P = -\frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 - \frac{1}{2} \sin^2 \varphi_0 \sin \vartheta_0 \\ - 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 + 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0$$

$$J = \frac{1}{2 \sin \mu_0}$$

$$K = -\frac{1}{4 \sin^2 \mu_0} \left[ \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 \{ \sin^2(2\vartheta_0 + \varphi_0) - \sin^2(\vartheta_0 + \varphi_0) \} \right. \\ \left. - 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) + 2 \cos(2\vartheta_0 + \varphi_0) \sin \mu_0 \right]$$

M will drop out of the equation, therefore, it has not been evaluated.

$$R = \sin \vartheta_0$$

$$S = \cos \vartheta_0 [\sin(2\vartheta_0 + \varphi_0) + 2 \sin \mu_0]$$

$$T = \left[ -\frac{1}{2} \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \sin \vartheta_0 \cos^2 \varphi_0 - \frac{1}{2} \sin \vartheta_0 \sin^2 \varphi_0 \right. \\ \left. + 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \varphi_0 - 4 \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \right. \\ \left. + 2 \sin \vartheta_0 \sin \varphi_0 \sin \mu_0 + 2 \cos \vartheta_0 \cos \varphi_0 \sin \mu_0 \right. \\ \left. - 8 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \mu_0 - 2 \sin \vartheta_0 \sin^2 \mu_0 \right]$$

Let us now evaluate the various terms in equation 89.

$$(E - R) = 0 \quad ; \quad (A - C) = 0 \quad \text{and} \quad (EA - RC) = 0.$$

$$J(F - S) = -\cos \vartheta_0$$

$$K(F - S) = \frac{\cos \vartheta_0}{2 \sin \mu_0} \left[ \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 \{ \sin^2(2\vartheta_0 + \varphi_0) - \sin^2(\vartheta_0 + \varphi_0) \} \right. \\ \left. - 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) + 2 \cos(2\vartheta_0 + \varphi_0) \sin \mu_0 \right]$$

$$0 = -2 \cos(\theta + \phi) \cos \psi$$

$$0 = \frac{1}{2} \cos^2 \psi - \frac{1}{2} \cos^2(\psi + \theta) + \frac{1}{2} \cos^2(\psi + \phi) + \frac{1}{2} \cos^2(\psi + \theta + \phi)$$

$$0 = 2 \sin \psi$$

$$0 = 2 \sin(\psi + \theta) \cos \psi$$

$$0 = \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi)$$

$$0 = \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi)$$

$$\frac{1}{2 \sin \psi} = 0$$

$$0 = \frac{1}{2 \sin \psi} \left[ \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi) \right]$$

Will drop out of the equation, therefore, it has not been evaluated

$$0 = 2 \sin \psi$$

$$0 = \cos \psi [2 \sin(\psi + \theta) + 2 \sin \psi]$$

$$0 = \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi)$$

$$0 = \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi)$$

$$0 = \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi)$$

$$0 = \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi)$$

Let us now evaluate the various terms in equation 88

$$(E-R)=0; (A-C)=0 \text{ and } (E-A-R)=0$$

$$0 = -2 \cos \psi$$

$$K(F-2) = \frac{1}{2 \sin \psi} \left[ \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi) \right]$$

$$0 = \frac{1}{2 \sin \psi} \left[ \frac{1}{2} \sin^2 \psi - \frac{1}{2} \sin^2(\psi + \theta) + \frac{1}{2} \sin^2(\psi + \phi) + \frac{1}{2} \sin^2(\psi + \theta + \phi) \right]$$



$$J(P-T) = \frac{1}{2 \sin \mu_0} \left[ -\frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 - \frac{1}{2} \sin^2 \varphi_0 \sin \vartheta_0 \right. \\
- 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 + 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \\
+ \frac{1}{2} \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 - \frac{1}{2} \sin \vartheta_0 \cos^2 \varphi_0 + \frac{1}{2} \sin \vartheta_0 \sin^2 \varphi_0 \\
- 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \varphi_0 + 4 \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \\
- 2 \sin \vartheta_0 \sin \varphi_0 \sin \mu_0 - 2 \cos \vartheta_0 \cos \varphi_0 \sin \mu_0 \\
\left. + 8 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \mu_0 + 2 \sin \vartheta_0 \sin^2 \mu_0 \right].$$

The latter expression reduces to:

$$J(P-T) = \frac{1}{2 \sin \mu_0} \left[ -\frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \sin \vartheta_0 \sin^2 \varphi_0 \right. \\
+ 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 \\
- 2 \sin \mu_0 \{ \sin(2\vartheta_0 + \varphi_0) \sin \vartheta_0 - \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \\
\left. + 2 \sin \vartheta_0 \sin^2 \mu_0 \right].$$

$$J(BE-RD) = JE(B-D) \\
= \sin(2\vartheta_0 + \varphi_0) \sin \vartheta_0.$$

$$J(FA-SC) = JA(F-S) \\
= 2 \cos(\vartheta_0 + \varphi_0) \cos^2 \vartheta_0.$$

Having evaluated all the terms appearing in equation 89, let us substitute them in that equation.

$$N_3 \lambda_0 = \frac{LG}{\epsilon} \left[ -\frac{\nu}{\epsilon} \cos \vartheta_0 + \frac{\nu^2}{\epsilon^2} \left( \frac{1}{\sin \mu_0} \left\{ \frac{1}{4} \sin \vartheta_0 \cos^2 \vartheta_0 \sin^2(2\vartheta_0 + \varphi_0) \right. \right. \right. \\
- \frac{1}{4} \sin \vartheta_0 \cos^2 \vartheta_0 \sin^2(\vartheta_0 + \varphi_0) - 2 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \\
- \frac{1}{4} \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{4} \sin \vartheta_0 \sin^2 \varphi_0 \\
+ 2 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 + \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 \} \\
+ \{ \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 - \sin(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \\
+ \sin(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + 2 \cos(\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 \} \\
\left. + \sin \mu_0 \sin \vartheta_0 \right) + \dots \Big] + HG \left[ 1 + \frac{\nu}{\epsilon} C + \frac{\nu^2}{\epsilon^2} D + \dots \right].$$

Upon simplification, this becomes:

$$N_3 \lambda_0 = \frac{LG}{\epsilon} \left[ -\frac{\nu}{\epsilon} \cos \vartheta_0 + \frac{\nu^2}{\epsilon^2} \left( \frac{1}{\sin \mu_0} \left\{ -\frac{1}{4} \sin \vartheta_0 \cos^2 \vartheta_0 \sin^2(\vartheta_0 + \varphi_0) - \frac{1}{4} \sin^2(2\vartheta_0 + \varphi_0) \sin^3 \vartheta_0 \right. \right. \right. \\
+ \frac{1}{4} \sin \vartheta_0 \sin^2 \varphi_0 + \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 \} \\
+ \{ 2 \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 + 2 \cos(\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 \} \\
\left. + \sin \mu_0 \sin \vartheta_0 \right) + \dots \Big] + HG \left[ 1 + \frac{\nu}{\epsilon} C + \frac{\nu^2}{\epsilon^2} D + \dots \right].$$

Eqn. (90)





Let us now evaluate LG. Referring to figure I we see that:

$$LG = RN + GN - RL. \quad \text{Eqn. (91)}$$

In the right triangle HNR:

$$\begin{aligned} RN &= HR \sin\left(\frac{\pi}{2} - \vartheta - \varphi\right) \\ &= HR \cos(\vartheta + \varphi). \end{aligned}$$

Also, in the right triangle HNG

$$GN = HR \cos\left(\frac{\pi}{2} - \vartheta - \varphi\right) \tan i_3.$$

Substitute for  $i_3$  its value  $\left(\frac{\pi}{2} - \theta - \sigma - i_2'\right)$ . This yields:

$$GN = HR \sin(\vartheta + \varphi) \cot n(\vartheta' + \sigma + i_2')$$

Looking at the line AHR, we see that :

$$\begin{aligned} HR &= v(t_1'' + t_2'' - t_1' - t_2') \\ &= \frac{v}{c} (D_1'' + D_2'' - D_1' - D_2'). \end{aligned}$$

Let us replace HR by this value in the equations for RN and GN.

$$RN = \frac{v}{c} (D_1'' + D_2'' - D_1' - D_2') \cos(\vartheta + \varphi)$$

and

$$GN = \frac{v}{c} (D_1'' + D_2'' - D_1' - D_2') \sin(\vartheta + \varphi) \cot n(\vartheta' + \sigma + i_2').$$

To evaluate RL, we observe that:

$$RL = BL - BR$$

In triangle BLP we have, by the sine law:

$$\frac{\sin(j_1' - \omega)}{BL} = \frac{\sin(\pi - \vartheta)}{ct_2''}$$

or

$$BL = ct_2'' \frac{\sin(j_1' - \omega)}{\sin \vartheta}.$$

Also, in triangle BFR:





$$\frac{\sin \varphi}{B R} = \frac{\sin \vartheta}{v t_2''}$$

or

$$B R = v t_2'' \frac{\sin \varphi}{\sin \vartheta}$$

Therefore:

$$R L = c t_2'' \frac{\sin(j_1' - \omega)}{\sin \vartheta} - v t_2'' \frac{\sin \varphi}{\sin \vartheta}$$

or

$$R L = D_2'' \frac{[\sin(j_1' - \omega) - \frac{v}{c} \sin \varphi]}{\sin \vartheta}$$

Now, in equation 91 replace RN, GN and RL by their values just obtained.

Also observe that in equation 90 we need  $LG/\frac{v}{c}$ . Therefore:

$$\begin{aligned} \frac{L G}{\frac{v}{c}} = & (D_1'' + D_2'' - D_1' - D_2') [\cos(\vartheta + \varphi) + \sin(\vartheta + \varphi) \operatorname{ctn}(\vartheta' + \sigma + i_2')] \\ & - D_2'' \frac{[\sin(j_1' - \omega) - \frac{v}{c} \sin \varphi]}{\frac{v}{c} \sin \vartheta} \end{aligned}$$

Eqn. (92)

Again, observing equation 90 we see that  $LG/\frac{v}{c}$  is needed only up to the first power of  $v/c$ , since  $(v/c)^1$  is the lowest power of its coefficient.

$$\begin{aligned} & \cos(\vartheta + \varphi) + \sin(\vartheta + \varphi) \operatorname{ctn}(\vartheta' + \sigma + i_2') \\ & = \frac{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')}{\sin(\vartheta' + \sigma + i_2')} \end{aligned}$$

Substituting the value for  $\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')$  given on the top of page 58, we get:

$$\cos(\vartheta + \varphi) + \sin(\vartheta + \varphi) \operatorname{ctn}(\vartheta' + \sigma + i_2') = \frac{\sin(\vartheta + \vartheta' + \varphi)}{\frac{D_2'}{S'} \sin(\vartheta' + \sigma + i_2')}$$

$$\frac{ER}{\sin \theta} = \frac{\sin \theta}{\sin \theta}$$

$$ER = \sin \theta$$

Therefore:

$$RL = \frac{ER \sin \theta}{\sin \theta} = ER$$

$$RL = \frac{ER \sin \theta}{\sin \theta} = ER$$

Now, in equation (1) we have  $ER$ ,  $RL$  and  $ER$  by their values just obtained.

The objective of this section is to find  $ER$ . Therefore:

$$\frac{ER}{\sin \theta} = \frac{ER \sin \theta}{\sin \theta} = ER$$

Eq. (2)

Again, observing equation (2) we see that  $ER$  is needed only in the first term of the right hand side, since  $(\sin \theta)^2$  is the forward power of the coefficient.

$$\frac{ER \sin \theta}{\sin \theta} = \frac{ER \sin \theta}{\sin \theta}$$

Substituting the value for  $\sin \theta$  in equation (1) gives us the top of page 12.

we get:

$$\frac{ER \sin \theta}{\sin \theta} = \frac{ER \sin \theta}{\sin \theta}$$



From the equation at the bottom of page 59 we see that :

$$\frac{1}{\frac{D_2'}{S'}} = 1 + \frac{v}{c} \cos(2\theta_0 + \varphi_0) + \dots$$

Multiplying this latter expression by equations 7 and 46, we get:

$$\frac{\sin(\theta + \theta' + \varphi)}{\frac{D_2'}{S'} \sin(\theta' + \sigma + i_1)} = \sin(2\theta_0 + \varphi_0) \left[ \frac{1}{\sin \theta_0} - \frac{v}{c} \frac{1}{\sin^2 \theta_0} \{ \cos \theta_0 \sin(2\theta_0 + \varphi_0) - \cos(2\theta_0 + \varphi_0) \sin \theta_0 \} \right]$$

which reduces to:

$$\frac{\sin(\theta + \theta' + \varphi)}{\frac{D_2'}{S'} \sin(\theta' + \sigma + i_1)} = \sin(2\theta_0 + \varphi_0) \left[ \frac{1}{\sin \theta_0} - \frac{v}{c} \frac{\sin(\theta_0 + \varphi_0)}{\sin^2 \theta_0} + \dots \right].$$

Eqn. (93)

Subtracting equations 61 and 62 from equations 63 and 64, we obtain:

$$(D_1'' + D_2'' - D_1' - D_2') = 2(S_0'' - S_0') - 2 \frac{v}{c} S_0'' \frac{\cos \theta_0}{\sin \theta_0} \sin \mu_0 + \dots$$

Eqn. (94)

Let us now multiply equation 93 by equation 94. This gives:

$$\begin{aligned} (D_1'' + D_2'' - D_1' - D_2') [\cos(\theta + \varphi) + \sin(\theta + \varphi) \cot(\theta' + \sigma + i_1)] \\ = 2(S_0'' - S_0') \frac{\sin(2\theta_0 + \varphi_0)}{\sin \theta_0} \\ - \frac{v}{c} \frac{1}{\sin^2 \theta_0} \{ 2(S_0'' - S_0') \sin(2\theta_0 + \varphi_0) \sin(\theta_0 + \varphi_0) + 2S_0'' \sin(2\theta_0 + \varphi_0) \cos \theta_0 \sin \mu_0 \} + \dots \end{aligned}$$

Eqn. (95)

We must now evaluate :

$$D_2'' \frac{[\sin(j_1' - \omega) - \frac{v}{c} \sin \varphi]}{\frac{v}{c} \sin \theta}$$

First, we shall treat  $[\sin(j_1' - \omega) - v/c \sin \theta]$ .

From the equation at the bottom of page 28 we have that :

$$\frac{1}{D_2} = 1 + \frac{v}{c} \cos(\theta_0 + \theta_0')$$

Substituting this latter expression by equations 7 and 10, we get:

$$\frac{D_1}{2} \sin(\theta' + \theta_0') = \frac{2\pi N(\theta + \theta_0)}{2\pi N(\theta_0 + \theta_0')} \left[ \frac{1}{\sin \theta_0} - \frac{v}{c} \frac{1}{\sin \theta_0} \cos(\theta_0 + \theta_0') \right]$$

which reduces to:

$$\frac{D_1}{2} \sin(\theta' + \theta_0') = \frac{2\pi N(\theta + \theta_0)}{2\pi N(\theta_0 + \theta_0')} \left[ \frac{1}{\sin \theta_0} - \frac{v}{c} \frac{1}{\sin \theta_0} \cos(\theta_0 + \theta_0') \right] + \dots$$

Eqn. (22)

Substituting equations 61 and 62 into equations 22 and 24, we

obtain:

$$D_1'' + D_1' - D_1 - D_1' = 2(2'' - 2') - 2 \frac{v}{c} 2'' \frac{\cos \theta_0}{\sin \theta_0} \sin \theta_0 + \dots$$

Eqn. (24)

Eqn. (24) is now easily solved by equation 22. This gives:

$$D_1'' + D_1' - D_1 - D_1' = \frac{2(2'' - 2')}{\sin \theta_0} \left[ \cos(\theta_0 + \theta_0') + \sin(\theta_0 + \theta_0') \right]$$

Eqn. (25)

We must now evaluate :

$$D_1'' = \frac{2\pi N(\theta + \theta_0)}{2\pi N(\theta_0 + \theta_0')} \left[ \frac{1}{\sin \theta_0} - \frac{v}{c} \frac{1}{\sin \theta_0} \cos(\theta_0 + \theta_0') \right]$$

First, we shall find  $\sin(\theta_0 + \theta_0')$  :



$$\sin(j'_1 - \omega) = \sin j'_1 \cos \omega - \cos j'_1 \sin \omega$$

Substitute equations 4, 17, 18, 49 and 50.

$$\begin{aligned} \sin(j'_1 - \omega) - \frac{v}{c} \sin \varphi &= \frac{v}{c} (\sin \varphi_0 - \sin \mu_0) + \frac{v^2}{c^2} \cos \varphi_0 (\sin \varphi_0 - 2 \sin \mu_0) \\ &\quad - \frac{v}{c} \sin \mu_0 + \frac{v^2}{c^2} \sin \varphi_0 \cos \varphi_0 - \frac{v}{c} \sin \varphi_0. \end{aligned}$$

Collecting terms, we obtain:

$$\sin(j'_1 - \omega) - \frac{v}{c} \sin \varphi = -2 \frac{v}{c} \sin \mu_0 + 2 \frac{v^2}{c^2} \cos \varphi_0 (\sin \varphi_0 - \sin \mu_0) + \dots$$

Cancelling a  $v/c$  and substituting equation 9 for  $\sin \vartheta$  and equation 64 for  $D_2''$  yields:

$$\begin{aligned} D_2'' \frac{[\sin(j'_1 - \omega) - \frac{v}{c} \sin \varphi]}{\frac{v}{c} \sin \vartheta} &= \frac{S_0''}{\sin \vartheta_0} \left[ 1 + \frac{v}{\sin \vartheta_0} \{ \sin \vartheta_0 \cos \varphi_0 + 2 \cos \vartheta_0 \sin \mu_0 \} \right] \\ &\quad [-2 \sin \mu_0 + 2 \frac{v}{c} \cos \varphi_0 (\sin \varphi_0 - \sin \mu_0)]. \end{aligned}$$

Multiplying and collecting terms, reduces the latter to:

$$\begin{aligned} D_2'' \frac{[\sin(j'_1 - \omega) - \frac{v}{c} \sin \varphi]}{\frac{v}{c} \sin \vartheta} &= \frac{2 S_0''}{\sin \vartheta_0} [-\sin \mu_0 + \\ &\quad + \frac{v}{c} \frac{1}{\sin \vartheta_0} \{ \sin \vartheta_0 \sin \varphi_0 \cos \varphi_0 + 2 \cos \vartheta_0 \sin^2 \mu_0 \} + \dots]. \end{aligned}$$

Subtracting this latter expression from equation 95, we obtain:

$$\begin{aligned} \frac{L G}{c} &= 2(S_0'' - S_0') \frac{\sin(2\vartheta_0 + \varphi_0)}{\sin \vartheta_0} + 2 S_0'' \frac{\sin \mu_0}{\sin \vartheta_0} \\ &\quad + \frac{v}{c} \frac{1}{\sin^2 \vartheta_0} [-2(S_0'' - S_0') \sin(2\vartheta_0 + \varphi_0) \sin(\vartheta_0 + \varphi_0) \\ &\quad - 2 S_0'' \{ \sin(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \sin \mu_0 + \sin \vartheta_0 \cos \varphi_0 \sin \varphi_0 \\ &\quad + 2 \cos \vartheta_0 \sin^2 \mu_0 \} + \dots. \end{aligned} \quad \text{Eqn. (96)}$$

To evaluate HG, consider the right triangle HNG.

To evaluate  $H_0$ , consider the right triangle  $H_0$ .

Eqn. (28)

$$\begin{aligned}
 & + 2 \cos \theta_0 \sin^2 \theta_0 \sin^2 \theta_0 \\
 & - 2 \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \} \\
 & + \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \} \\
 & \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \} \\
 & \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \}
 \end{aligned}$$

Substituting this latter expression from equation 25, we obtain:

$$\begin{aligned}
 & \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \} \\
 & \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \} \\
 & \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \}
 \end{aligned}$$

Multiplying and collecting terms, reduce the latter to:

$$\begin{aligned}
 & \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \} \\
 & \frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \}
 \end{aligned}$$

where:

Canceling  $\sqrt{2}$  and substituting equation 2 for  $\sin \theta_0$  and equation 25 for  $\cos^2 \theta_0$

$$\frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \}$$

Collecting terms, we obtain:

$$\frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \}$$

$$\frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \}$$

Substituting equations 2, 25, 26, 27 and 28.

$$\frac{1}{2} \sin^2 \theta_0 \{ \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 + \sin^2 \theta_0 \cos^2 \theta_0 \}$$



$$\begin{aligned}
 HG &= \frac{NG}{\sin i_3} \\
 &= \frac{\frac{v}{c}(D_1'' + D_2'' - D_1' - D_2') \sin(\vartheta + \varphi) \operatorname{ctn}(\vartheta' + \sigma + i_2')}{\sin(\frac{\pi}{2} - \vartheta' - \sigma - i_2')}
 \end{aligned}$$

Simplifying, we obtain:

$$HG = \frac{\frac{v}{c}(D_1'' + D_2'' - D_1' - D_2') \sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')}$$

Substitute equations 5, 46 and 94.

$$\begin{aligned}
 HG &= \frac{v}{c} \sin(\vartheta_0 + \varphi_0) \left[ 2(s_0'' - s_0') \frac{1}{\sin \vartheta_0} \right. \\
 &\quad \left. - 2 \frac{v}{c} \frac{1}{\sin^2 \vartheta_0} \{ (s_0'' - s_0') \sin(2\vartheta_0 + \varphi_0) \cos \vartheta_0 + s_0'' \cos \vartheta_0 \sin \mu_0 \} + \dots \right].
 \end{aligned}$$

Substituting the value of C and D in equation 71, we obtain:

$$\begin{aligned}
 HG(1 + \frac{v}{c}C + \frac{v^2}{c^2}D) &= 2 \frac{v}{c} (s_0'' - s_0') \frac{\sin(\vartheta_0 + \varphi_0)}{\sin \vartheta_0} \\
 &\quad + 2 \frac{v^2}{c^2} \frac{\sin(\vartheta_0 + \varphi_0)}{\sin^2 \vartheta_0} \{ - (s_0'' - s_0') [2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \sin \vartheta_0 + \sin(2\vartheta_0 + \varphi_0) \cos \vartheta_0] \\
 &\quad - s_0'' \cos \vartheta_0 \sin \mu_0 \} + \dots
 \end{aligned}$$

Eqn. (97)

Now, in equation 90, substitute equation 96 for  $LG/\frac{v}{c}$  and multiply.

Then replace  $HG(1 + \frac{v}{c}C + \frac{v^2}{c^2}D)$  by equation 97. Arranging according to powers of  $v/c$ , we obtain:

$$H_0 = \frac{W_0}{2\pi\omega}$$

$$= \frac{\frac{1}{2}(D_1 + D_2 - D_3 - D_4) \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2)}{2\pi\omega(\frac{1}{2} - \theta_1 - \theta_2 - \frac{\pi}{2})}$$

Substituting, we obtain:

$$H_0 = \frac{\frac{1}{2}(D_1 + D_2 - D_3 - D_4) \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2)}{2\pi\omega(\frac{1}{2} - \theta_1 - \theta_2 - \frac{\pi}{2})}$$

Substituting equations 9, 10 and 11:

$$H_0 = \frac{1}{2\pi\omega} \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) \left[ \frac{1}{2}(\theta_1'' - \theta_2'') - \frac{1}{2}(\theta_1' - \theta_2') \right]$$

$$= \frac{1}{2\pi\omega} \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) \left[ \frac{1}{2}(\theta_1'' - \theta_2'') - \frac{1}{2}(\theta_1' - \theta_2') \right]$$

Substituting the value of  $\theta_1$  and  $\theta_2$  in equation 11, we obtain:

$$H_0(1 + \frac{1}{2}C + \frac{1}{2}D) = \frac{1}{2\pi\omega} \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) \left[ \frac{1}{2}(\theta_1'' - \theta_2'') - \frac{1}{2}(\theta_1' - \theta_2') \right]$$

$$+ \frac{1}{2\pi\omega} \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) \left[ \frac{1}{2}(\theta_1'' - \theta_2'') - \frac{1}{2}(\theta_1' - \theta_2') \right]$$

$$= \frac{1}{2\pi\omega} \sin(\theta_1 + \theta_2) \cos(\theta_1 + \theta_2) \left[ \frac{1}{2}(\theta_1'' - \theta_2'') - \frac{1}{2}(\theta_1' - \theta_2') \right]$$

Eqn. (12)

Now, in equation 12, substitute equation 9 for  $\theta_1$  and multiply:

The resulting  $W_0(1 + \frac{1}{2}C + \frac{1}{2}D)$  is equation 12. Assuming according to power

of  $\omega$ , we obtain:



$$\begin{aligned}
N_3 \lambda_0 = & -\frac{v}{c} [2(s_0'' - s_0') \frac{1}{\sin \vartheta_0} \{ \sin(2\vartheta_0 + \varphi_0) \cos \vartheta_0 - \sin(\vartheta_0 + \varphi_0) \} + 2s_0'' \frac{\cos \vartheta_0}{\sin \vartheta_0} \sin \mu_0] \\
& - \frac{v^2}{c^2} \frac{1}{\sin^2 \vartheta_0} [ -2(s_0'' - s_0') \{ \sin(2\vartheta_0 + \varphi_0) \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \\
& - 2 \cos(\vartheta_0 + \varphi_0) \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 - \sin(2\vartheta_0 + \varphi_0) \sin(\vartheta_0 + \varphi_0) \cos \vartheta_0 \\
& + 2 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \\
& + 2 \sin(2\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \} \\
& - \frac{\sin(2\vartheta_0 + \varphi_0)}{\sin \mu_0} \{ \frac{1}{4} \sin^2 \vartheta_0 \cos^2 \vartheta_0 \sin^2(\vartheta_0 + \varphi_0) \\
& + \frac{1}{4} \sin^2(2\vartheta_0 + \varphi_0) \sin^4 \vartheta_0 - \frac{1}{4} \sin^2 \vartheta_0 \sin^2 \varphi_0 \\
& - \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^3 \vartheta_0 \cos \vartheta_0 \\
& + \sin \mu_0 \sin(2\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \} \\
& - 2s_0'' \{ \sin \vartheta_0 \cos \vartheta_0 \sin \varphi_0 \cos \varphi_0 - \frac{1}{4} \sin^2 \vartheta_0 \cos^2 \vartheta_0 \sin^2(\vartheta_0 + \varphi_0) \\
& - \frac{1}{4} \sin^2(2\vartheta_0 + \varphi_0) \sin^4 \vartheta_0 + \frac{1}{4} \sin^2 \vartheta_0 \sin^2 \varphi_0 \\
& + \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^3 \vartheta_0 \cos \vartheta_0 \\
& + \sin \mu_0 \{ \sin(2\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 + 2 \cos(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \\
& + 2 \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \} \\
& + \sin^2 \mu_0 \{ 2 \cos^2 \vartheta_0 + \sin^2 \vartheta_0 \} ] + \dots
\end{aligned}$$

Upon simplifications, this becomes:

$$\begin{aligned}
N_3 \lambda_0 = & -\frac{v}{c} [2(s_0'' - s_0') \cos(2\vartheta_0 + \varphi_0) + 2s_0'' \frac{\cos \vartheta_0}{\sin \vartheta_0} \sin \mu_0] \\
& - \frac{v^2}{c^2} \frac{1}{\sin^2 \vartheta_0} [ -2(s_0'' - s_0') \{ 2 \cos(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \{ \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \sin(2\vartheta_0 + \varphi_0) \} \\
& - \frac{\sin(2\vartheta_0 + \varphi_0)}{\sin \mu_0} \{ \frac{1}{4} \sin^2 \vartheta_0 \cos^2 \vartheta_0 [\sin^2(\vartheta_0 + \varphi_0) - \sin^2(2\vartheta_0 + \varphi_0)] \} \\
& + \sin \mu_0 \sin(2\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \} \\
& - 2s_0'' \{ \sin \vartheta_0 \cos \vartheta_0 \{ \sin \varphi_0 \cos \varphi_0 \\
& + \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 [\sin^2(2\vartheta_0 + \varphi_0) - \sin^2(\vartheta_0 + \varphi_0)] \} \\
& + \sin \mu_0 \sin \vartheta_0 \cos \vartheta_0 \{ 3 \cos(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0 \} \\
& + \sin^2 \mu_0 (1 + \cos^2 \vartheta_0) \} ] + \dots
\end{aligned}$$

Eqn. (98)





TOTAL DIFFERENCE OF NUMBER OF WAVE LENGTHS IN THE TWO PATHS

To find the total difference of the number of wave lengths in the two paths, it is necessary to subtract equation 98 from equation 85. Let:

$$N_1 + N_2 - N_3 = N$$

Then:

$$\begin{aligned}
 N \lambda_0 = & 2(S_0'' - S_0') + \frac{v}{c} [-2(S_0'' - S_0') \{ \cos \varphi_0 - \cos(2\vartheta_0 + \varphi_0) \}] \\
 & + \frac{v^2}{c^2} \frac{1}{\sin^2 \vartheta_0} \left[ 2(S_0'' - S_0') \left\{ \frac{1}{2} \sin^2 \vartheta_0 + \frac{3}{2} \sin^2 \vartheta_0 \cos^2 \varphi_0 \right. \right. \\
 & \quad - 2 \cos(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 [\cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \sin(2\vartheta_0 + \varphi_0)] \\
 & \quad + \frac{\sin(2\vartheta_0 + \varphi_0)}{\sin \mu_0} \left[ \frac{1}{4} \sin^2 \vartheta_0 \cos^2 \vartheta_0 (\sin^2(\vartheta_0 + \varphi_0) - \sin^2(2\vartheta_0 + \varphi_0)) \right. \\
 & \quad \quad \left. \left. - \sin \mu_0 \sin(2\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \right\} \right. \\
 & \quad + 2S_0'' \{ \sin \mu_0 \sin \vartheta_0 [2 \cos(\vartheta_0 + \varphi_0) + \cos \vartheta_0 \cos \varphi_0] + \sin^2 \vartheta_0 \cos^2 \varphi_0 \\
 & \quad \quad - \sin \vartheta_0 \cos \vartheta_0 [\sin \varphi_0 \cos \varphi_0 \\
 & \quad \quad \quad \left. + \frac{1}{4} \sin \vartheta_0 \cos \vartheta_0 (\sin^2(2\vartheta_0 + \varphi_0) - \sin^2(\vartheta_0 + \varphi_0))] \} \\
 & \quad - \sin \mu_0 \sin \vartheta_0 \cos \vartheta_0 [3 \cos(2\vartheta_0 + \varphi_0) + 2 \cos(\vartheta_0 + \varphi_0) \cos \vartheta_0] \} \\
 & \quad \left. - S_0' \cos^2(2\vartheta_0 + \varphi_0) \right] + \dots \right.
 \end{aligned}$$

Eqn. (99)

The latter is the general equation of the difference of the number of wave lengths in the two paths, in terms of  $\vartheta_0$ ,  $\varphi_0$ ,  $\mu_0$ ,  $S_0''$ ,  $S_0'$ ,  $\lambda_0$ ,  $v$ , and  $c$ .



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# APPLICATION OF THE FINAL RESULT TO THE WORK OF DAYTON C. MILLER

In the work of D. C. Miller<sup>1</sup> the two arms were made equal. Also, since the two arms were at right angles to each other  $\theta_0$  was equal to  $45^\circ$ . Substituting these conditions in equation 99, we obtain:

$$S_0'' - S_0' = 0$$

and

$$\sin \theta_0 = \cos \theta_0 = \frac{1}{\sqrt{2}}$$

Let  $S_0''$  and  $S_0'$  be designated as  $S_0$ . Then:

$$N = \frac{4 S_0}{\lambda_0} \frac{v^2}{c^2} [\sin \mu_0 (\cos \varphi_0 + \sin \varphi_0) + \frac{15}{32} (\cos^2 \varphi_0 - \sin^2 \varphi_0) - \frac{7}{16} \sin \varphi_0 \cos \varphi_0] + \dots$$

In the latter equation let us substitute  $\sin \omega_0$  for  $v/c \sin \mu_0$  (cf. page 29). This gives:

$$N = \frac{4 S_0}{\lambda_0} \frac{v}{c} \sin \omega_0 (\cos \varphi_0 + \sin \varphi_0) + \frac{2 S_0}{\lambda_0} \frac{v^2}{c^2} \left[ \frac{15}{16} \cos 2 \varphi_0 - \frac{7}{16} \sin 2 \varphi_0 \right] + \dots$$

On page 227 of the article mentioned above, we find that:

$$\frac{2 S_0}{\lambda_0} = 1.12 \times 10^8$$

Also, on page 238 of that same article, we find that  $\omega_0$  was equal to  $4''$ .

Therefore:

$$\sin \omega_0 = 2 \times 10^{-5}$$

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1. Miller, D. C., Rev. Mod. Phys., 5, 203, (1933).

# APPLICATION OF THE TRIANGLE RULE TO THE WORK OF DAYTON C. MILLER

In the work of D. C. Miller, the two sides were made equal. Also, since the two sides were at right angles to each other, the angle between them was equal to 90°. Substituting these angles in equation (1), we obtain:

$$S_1^2 - S_2^2 = 0$$

and

$$\sin \theta_1 = \cos \theta_2 = \frac{1}{\sqrt{2}}$$

Let  $S_1$  and  $S_2$  be designated as  $S$ . Then:

$$H = \frac{4S}{\lambda} \left[ \frac{1}{2} (\cos^2 \theta_1 + \sin^2 \theta_1) + \frac{1}{2} (\cos^2 \theta_2 + \sin^2 \theta_2) - \frac{1}{2} \sin 2\theta_1 \cos 2\theta_2 \right]$$

In the latter equation let us substitute also for  $\sin 2\theta_1$ .

Let us have  $\sin 2\theta_1$ . This gives:

$$H = \frac{4S}{\lambda} \left[ \frac{1}{2} (\cos^2 \theta_1 + \sin^2 \theta_1) + \frac{1}{2} (\cos^2 \theta_2 + \sin^2 \theta_2) - \frac{1}{2} \sin 2\theta_1 \right]$$

We have 287 of the article mentioned above, we find that:

$$\frac{2S}{\lambda} = 1.15 \times 10^8$$

Also, on page 238 of that same article, we find that  $\theta_1$  was equal to 45°.

Therefore:

$$\sin 2\theta_1 = \sin 90^\circ = 1$$



Let us substitute these values in the equation for N. This gives:

$$N = \frac{v}{c} 4.48 \sqrt{2} \times 10^3 \sin\left(\frac{\pi}{4} + \varphi_0\right) + \frac{v^2}{c^2} [1.05 \times 10^8 \cos 2\varphi_0 - .49 \times 10^8 \sin 2\varphi_0] + \dots$$

Let:

$$A = 1.05 \times 10^8$$

and

$$B = .49 \times 10^8$$

Also, let:

$$\begin{aligned} A \cos 2\varphi_0 - B \sin 2\varphi_0 &= C \sin(2\varphi_0 + \alpha_0) \\ &= C \sin 2\varphi_0 \cos \alpha_0 + C \cos 2\varphi_0 \sin \alpha_0 \end{aligned}$$

Equating the coefficients of  $\sin 2\varphi_0$ , we get:

$$B = -C \cos \alpha_0$$

Also, equating the coefficients of  $\cos 2\varphi_0$ , we get:

$$A = C \sin \alpha_0$$

Dividing the latter by the former, we get:

$$\begin{aligned} \tan \alpha_0 &= -\frac{A}{B} \\ &= -2.14 \end{aligned}$$

Therefore:

$$\alpha_0 = 115^\circ 00'$$

Thus:

$$C = \frac{A}{\sin \alpha_0} = \frac{1.05 \times 10^8}{.9063} = 1.16 \times 10^8$$

Substituting these values in the equation for N, we obtain:

$$N = \frac{v}{c} 6.34 \times 10^3 \sin\left(\frac{\pi}{4} + \varphi_0\right) + \frac{v^2}{c^2} [1.16 \sin(2\varphi_0 + 115^\circ 00')] 10^8$$

Eqn. (100)

Substituting these values in the equation for  $x$ , we get:

$$M = \frac{1}{2} \cdot 4.48 \times 10^3 \sin\left(\frac{\pi}{4} + 4.4\right) + \frac{1}{2} (1.02 \times 10^3 \cos 4.4 - 4.4 \times 10^3 \sin 4.4)$$

$$A = 1.02 \times 10^3$$

$$B = 4.4 \times 10^3$$

$$A \cos 2\phi - B \sin 2\phi = C \cos(2\phi + \phi_0)$$

$$= C \cos 2\phi \cos \phi_0 - C \sin 2\phi \sin \phi_0$$

Equating the coefficients of  $\sin 2\phi$ , we get:

$$B = -C \sin \phi_0$$

Also, equating the coefficients of  $\cos 2\phi$ , we get:

$$A = C \cos \phi_0$$

Dividing the latter equation by the former, we get:

$$\tan \phi_0 = -\frac{B}{A}$$

$$\phi_0 = -2.14$$

Therefore:

$$\phi_0 = 113.00^\circ$$

Thus:

$$C = \frac{A}{\cos \phi_0} = \frac{1.02 \times 10^3}{\cos 113.00^\circ} = 2.16 \times 10^3$$

Substituting these values in the equation for  $x$ , we obtain:

$$M = \frac{1}{2} \cdot 2.16 \times 10^3 \sin\left(\frac{\pi}{4} + 4.4\right) + \frac{1}{2} (2.16 \times 10^3 \cos(4.4 + 113.00^\circ)) / 10^2$$



Finally, it will be desirable to plot the curves of the first and second order terms of equation 100 in order to compare them with the experimental curves obtained by D. C. Miller. So that the curves of the first and second order terms may be comparable, let us assume that  $v/c$  equals  $10^{-4}$ .

Table of Values of First and Second Order

Terms for Various Values of  $\phi_0$

$\phi_0$	$\frac{v}{c} 6.34 \times 10^3 \sin(\frac{\pi}{4} + \phi_0)$	$\frac{v^2}{c^2} 1.16 \times 10^8 \sin(2\phi_0 + 115^\circ 00')$
$0^\circ$	0.448	1.05
$32^\circ 30'$	0.00	0.00
$45^\circ$	0.634 Max.	-0.49
$77^\circ 30'$	.	-1.16 Min.
$90^\circ$	0.448	-1.05
$122^\circ 30'$	.	0.00
$135^\circ$	0.000	0.49
$167^\circ 30'$	.	1.16 Max.
$180^\circ$	-0.448	1.05
$212^\circ 30'$	.	0.00
$225^\circ$	-0.634 Min.	-0.49
$257^\circ 30'$	.	-1.16 Min.
$270^\circ$	-0.448	-1.05
$302^\circ 30'$	.	0.00
$315^\circ$	0.000	0.49
$347^\circ 30'$	.	1.16 Max.
$360^\circ$	0.448	1.05

Finally, it will be desirable to plot the curves of the first and second order terms of equation 100 in order to compare them with the experimental curves obtained by D. C. Miller. So that the curves of the first and second order terms may be comparable, let us assume that  $\gamma_c$  equals 10.

Table of Values of First and Second Order Terms for Various Values of  $\lambda$

$\Phi$	$\frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda^2} \right)$	$\frac{1}{2} \left( \frac{1}{\lambda} - \frac{1}{\lambda^2} \right)$
0°	0.448	1.00
30°	1.00	0.00
45°	0.634 Max.	-0.14
60°	0.448	-1.18 Min.
75°	0.448	-1.00
90°	0.000	0.00
105°	0.000	0.44
120°	0.448	1.18 Max.
135°	0.448	1.00
150°	0.634 Min.	-0.14
165°	0.448	-1.18 Min.
180°	0.000	0.00
195°	0.000	0.44
210°	0.448	1.18 Max.
225°	0.448	1.00



Variation of Number of Fringes with  $\phi$ .

A. First Order Component    B. Second Order Component

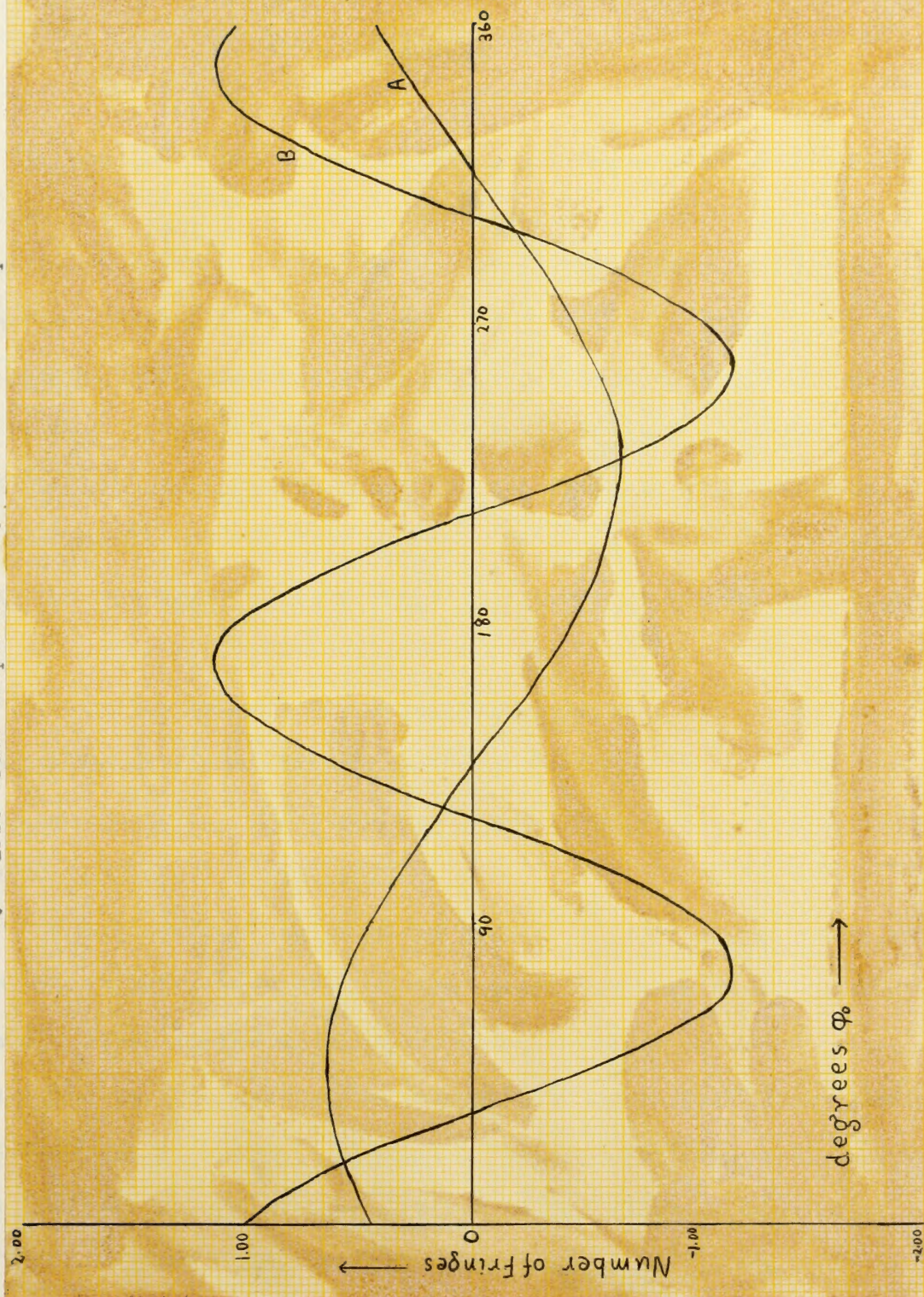


FIGURE VIII



11/10/1911

11/10/1911

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### CONCLUSION

On page 85 of this paper, we have a general equation of the difference of number of waves in the two paths traversed by the rays of light in a Michelson interferometer. According to this equation the actual fringe shift is a function of the effective difference in optical path, measured in terms of wave lengths along the various components of the paths from the time that the ray is first divided at the half silvered plate M until the rays have traveled their respective paths  $(D_1' + D_2' - D_3')$  and  $(D_1'' + D_2'' - D_3'')$ . A telescope equally inclined to the two final rays must be focused at the point from which the two rays appear to be coming. In figure I that point is designated by T.

In deriving equation 85, no assumption was made concerning the relative lengths of the two arms  $S_0'$  and  $S_0''$ . The only condition that need be imposed upon them is that  $S_0' - S_0''$  shall not exceed the limits of "coherence". Otherwise, interference is impossible.

The angle between the plate and either arm, that is,  $\theta_0$  is entirely arbitrary and is merely determined by the conditions given in any particular arrangement of the apparatus. In most cases this angle is made to equal  $45^\circ$ . The value of  $\theta_0$  is very important in determining the maximum fringe shift.

The angle  $\mu_0$  is also a constant of any given interferometer. Its value is fixed so that the fringes observed in the telescope shall be of convenient width. Upon substituting the proper values in the equations of the law of reflection from a moving mirror (equations 21 and 22) and applying the

## CONCLUSION

On page 85 of this paper, we have a general equation of the difference of number of waves for the two paths traversed by the rays of light in a Michelson interferometer. According to this equation the actual fringe shift is a function of the effective difference in optical path, measured in terms of wave lengths along the various components of the paths from the time that the ray is first divided to the time it is recombined. In order that the rays have traveled their respective paths  $(L_1 + \Delta L_1 - \Delta L_2)$  and  $(L_2 + \Delta L_2 - \Delta L_1)$ , a telescope equally inclined to the two arms must be focused at the point from which the two rays appear to be coming. In Figure 1 that point is designated by T.

In deriving equation 85, no assumption was made concerning the relative lengths of the two arms  $L_1$  and  $L_2$ . The only condition that need be imposed upon them is that  $L_1 - L_2$  shall not exceed the limits of "coherence". Otherwise, interference is impossible.

The angle between the paths and either arm, that is,  $\theta$ , is entirely arbitrary and is merely indicated by the conditions given in any particular arrangement of the apparatus. In any case this angle is made to equal 45°. The value of  $\theta$  is very important in determining the various fringe shifts.

The angle  $\phi_0$  is also important in the interferometer. The value of  $\phi_0$  is that the fringes observed in the telescope shall be of constant width. For angles other than  $\phi_0$  the values in the equations of the law of reflection from a moving mirror (equations 41 and 42) and applying the



Lorentz transformations it was found that these equations did not reduce to the simple law of reflection, namely that the angle of incidence is equal to the angle of reflection. This suggests that the angle of reflection depends upon the sidereal time. This conclusion is consistent with the results of E. Esclançon<sup>1</sup> and E. Cavallo<sup>2</sup>.

In order to obtain a curve of the number of fringes plotted against  $\phi_0$ , the constants of D. C. Miller's interferometer at Mount Wilson were substituted in equation 99. This led finally to equation 100. In this equation the first order term is periodic in a full turn of the instrument. Although the latter is a first order term, its magnitude is comparable with that of the second order term. The reason for this is that  $\sin \omega_0$ , which is a factor of the first order term is itself of the magnitude of  $v/c$ . Since, in substituting for  $\sin \mu_0$  equation 99, that term dropped to the first order; it might be argued that, had there been a third order term, the part of it that would have as its coefficient  $\sin \mu_0$ , would drop to the second term. However, since its magnitude would be comparable with that of the third order terms it is negligible to the order of approximation of this work.

In examining the first order term we see that when  $\phi_0 = 45^\circ$ , the fringe shift due to this component is a maximum on one side of the zero mark in the telescope and for  $\phi_0 = 225^\circ$  it is a maximum on the other side.

The second order term of equation 100 is doubly periodic upon a single rotation of the interferometer. The maxima due to this component are at  $\phi_0 = 167^\circ 30'$  and  $\phi_0 = 347^\circ 30'$  on one side of the zero mark in the teles-

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1. Esclançon, E., Journal des Observateurs 11, 49, (1928)

2. Cavallo, E., Comptes Rendus, 198, 247, (1934)





cope and at  $\phi_0 = 77^\circ 30'$  and  $\phi_0 = 257^\circ 30'$  on the other side of the zero mark.

On page 227 of the article by D. C. Miller, there are graphs, obtained by a harmonic analyzer, of the experimental curves showing the readings for four successive sets of observations made on April 2, 1925. Comparing those curves with the theoretical curves given on page 89 of this paper, we see that the latter set of curves is in form similar to those obtained by experiment.

In conclusion, we may say, therefore, that if the equations governing the Michelson interferometer are set up from the point of view of an observer fixed in the ether and are transformed by means of the Lorentz transformation the results required by theory are consistent with those obtained by experiment. Until now, it has generally been assumed that the theory required a zero effect contrary to that obtained by D. C. Miller.

It is the author's pleasure to acknowledge his indebtedness for the very valuable suggestions and criticisms received from Dr. Royal M. Frye, under whose direction this problem has been carried out.

copy was at  $\lambda = 11.30$  and  $\lambda = 11.35$  on the other side of the same curve.

On page 127 of the article by U. G. Miller, there are graphs showing

the variation of the refractive index of the experimental curves showing the variation of

the refractive index of the experimental curves showing the variation of the refractive index

curves with the theoretical curves given in page 12 of this paper, we see

that the latter set of curves is in better agreement with those obtained by experi-

ment.

In conclusion, we may say, therefore, that if the equations govern-

ing the Michelson interferometer are set up from the point of view of an

observer fixed in the ether and are transformed to a frame of reference trans-

formation the results obtained in theory are in excellent agreement with those obtained

by experiment. With this it is generally seen assumed that the theory re-

quires a new effort contrary to that obtained by U. G. Miller.

It is the authors pleasure to acknowledge his indebtedness for the

very valuable suggestions and criticisms received from Dr. Royal W. Ryce,

under whose direction this problem has been carried out.



### COMPREHENSIVE ABSTRACT

In this dissertation the general theory of the measurement of the ether drift by a Michelson interferometer is developed. The work is based upon three hypotheses which are widely used in physics. The first of these is that light is propagated as a wave disturbance according to Huygens' principle. The second hypothesis is that there exists a stationary ether through which light is propagated, and with respect to which the earth is moving with some relative velocity. The third hypothesis employed in this paper is one of the basic hypotheses of the theory of Relativity, namely that the velocity of light has a constant value relative to all inertial systems.

The interferometer contains a half-silvered mirror which partially transmits and partially reflects parallel light which is incident upon it. The transmitted and reflected rays are reflected at the end of their paths so as to return to the plate, where again one beam is reflected and the other is transmitted. Finally both rays are brought to a focus by a lens. The division and subsequent reuniting of a beam of light under these conditions brings about what are known as "interference fringes". If the hypotheses mentioned previously are correct, the position of these fringes should depend upon the orientation of the instrument with respect to the ether. The interference fringes themselves are due to the phase relationships existing between the wave fronts of the two reunited rays. These in turn are affected by the amount and direction of the motion of the interferometer during the interval of time that the beam of light is divided at the half-silvered plate and subsequently

# CONCLUSIONS

In this discussion the general theory of the measurement of the time interval by a Michelson interferometer is developed. The work is based upon three hypotheses which are widely used in physics. The first of these is that light is propagated as a wave disturbance according to Huygens' principle. The second hypothesis is that there exists a stationary ether through which light is propagated, and with respect to which the earth is moving with some relative velocity. The third hypothesis emphasized in this paper is one of the basic hypotheses of the theory of relativity, namely that the velocity of light has a constant value relative to all inertial systems.

The interferometer contains a half-silvered mirror which partially transmits and partially reflects parallel light which is incident upon it. The transmitted and reflected rays are reflected at the end of their paths so as to return to the plate, where again the beam is reflected and the other is transmitted. Finally both rays are brought to a focus by a lens. The division and subsequent reuniting of a beam of light under these conditions brings about what are known as "interference fringes". If the hypotheses mentioned are correctly and correctly, the position of these fringes should depend upon the orientation of the instrument with respect to the ether. The interference fringes themselves are due to the phase relationship existing between the two beams of the two reunited rays. There is thus no alteration of the time and direction of the motion of the interferometer during the interval of time that the beam of light is divided at the half-silvered mirror and subsequently



brought to focus by the viewing telescope. In consequence of this, as the interferometer is rotated the interference fringe will show slight displacement unless the axis of rotation of the instrument is parallel to the direction of the absolute velocity of the earth.

In the investigation, the path of each beam is carefully traced, from the time that it impinges upon the plate, until it enters the lens placed in its final path.

Due to the motion of the earth all distances, including the wave lengths of the light, are subjected to a contraction. This latter is a necessary consequence of the third hypothesis mentioned previously. The equation governing this contraction is the familiar Lorentz transformation equation:

$$x_b - x_a = (x_{0_b} - x_{0_a}) \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

where  $x_b$  and  $x_a$  are coordinates of a rod in a coordinate system fixed in the ether and  $x_{0_b}$  and  $x_{0_a}$  are the coordinates of a rod in a parallel moving coordinate system,  $v$  is the velocity of the moving system relative to the ether and  $c$  is the velocity of light. Angles and distances bearing the subscript zero will indicate those measured by the moving observer, all others will indicate these measured by the fixed observer.

Also, due to the motion of the earth, all angles undergo a change upon applying the Lorentz transformation. In the case of angles, two types must be dealt with. The first type includes angles of which one side is parallel to the direction of the motion. An equation characteristic of this type is:

$$\sin \varphi = \sin \varphi_0 \left(1 + \frac{1}{2} \frac{v^2}{c^2} \cos^2 \varphi_0 + \dots\right).$$

present to focus by the viewing telescope. In consequence of this, as the interferometer is rotated the interference fringes will show slight displacement unless the axis of rotation of the instrument is parallel to the direction of the absolute velocity of the earth.

In the investigation, the pairs of each beam is carefully focused from the time that it focuses upon the plate, until it returns the lens placed in the final path.

Due to the motion of the earth all distances, including the wave lengths of the light, are subjected to a contraction. This factor is a necessary consequence of the ether hypothesis mentioned previously. The equation governing this contraction is the familiar Lorentz transformation equation:

$$x' = \gamma(x - vt)$$

where  $x'$  and  $x$  are coordinates of a rod in a co-ordinate system fixed in the ether and  $x'_0$  and  $x_0$  are the coordinates of a rod in a parallel moving coordinate system,  $v$  is the velocity of the moving system relative to the ether and  $t$  is the velocity of light. As time and distance between the synchronizing light flashes are measured by the moving observer, all observers will find that these measured by the fixed observer.

Also, due to the motion of the earth, all angles undergo a change upon applying the Lorentz transformation. In the case of angles, two types must be dealt with. The first type includes angles of which one side is parallel to the direction of the motion. An equation characteristic of this type is:

$$\sin \theta' = \sin \theta \left( 1 + \frac{v}{c} \cos \theta \right)$$



where  $\phi$  is an angle one side of which is parallel to the direction of drift.

The second type of angles consists of those of which neither side is parallel to the direction of the drift. An equation characteristic of this type is:

$$\sin \theta = \sin \theta_0 \left[ 1 - \frac{1}{2} \frac{v^2}{c^2} \{ \sin^2(\theta_0 + \phi_0) - \cos^2 \phi_0 \} + \dots \right]$$

where  $\phi$  is the angle one side of which is parallel to the direction of the drift and  $\theta$  is an angle adjacent to its other side.

In consequence of the change of angles, the mirror of the apparatus that is set normal to the arm to which it is fixed, will not be normal as viewed by the observer fixed in the ether. This discrepancy was evaluated and taken into consideration. Similarly, the angle introduced between the normal to the second mirror and the arm to which it is fixed undergoes a modification. Likewise, this effect is considered.

It is well known that in the case of a moving mirror the angle of incidence is not equal to the angle of reflection. The equations governing this change were found to be:

$$\sin i' = \sin i - 2 \frac{v}{c} \sin \phi \sin i \cos i + 2 \frac{v^2}{c^2} \sin^2 \phi \sin i (2 \cos^2 i - 1) + \dots$$

and

$$\cos i' = \cos i + 2 \frac{v}{c} \sin \phi \sin^2 i - 4 \frac{v^2}{c^2} \sin^2 \phi \sin^2 i \cos i + \dots$$

where  $i'$  is the angle of reflection,  $i$  the angle of incidence and  $\phi$  is the angle between the direction of drift and the mirror. After substituting the proper values in the general equation, the Lorentz transformations were applied.

After determining the effect of the motion upon the angles involved in the path, the exact effect of the motion upon the length of each path is

where  $\theta$  is an angle one side of which is parallel to the direction of drift.  
 The second type of angles consists of those of which neither side  
 is parallel to the direction of the drift. An equation characterizing it  
 this type is:

$$\sin \theta = \sin \theta_0 \left[ 1 - \frac{1}{2} \frac{v}{c} \sin(2\theta_0) - \cos^2 \theta_0 \right]$$

where  $\theta_0$  is an angle one side of which is parallel to the direction of the  
 drift and the other side is perpendicular to it.  
 In comparison of the change of angles, the mirror of the apparatus

that is not parallel to the drift to which it is fixed, will not be rotated as  
 stated by the observer fixed in the ether. This discrepancy was explained  
 and taken into consideration. Similarly, the angle indicated between the  
 normal to the second mirror and the drift to which it is fixed undergoes a modification.  
 Therefore, this effect is not observed.

It is well known that in the case of a rotating mirror the angle of  
 reflection is not equal to the angle of incidence. The new theory explains  
 this change very simply as follows:

$$\sin \theta = \sin \theta_0 \left[ 1 - \frac{1}{2} \frac{v}{c} \sin(2\theta_0) + \frac{1}{2} \frac{v^2}{c^2} \sin^2 \theta_0 \right]$$

and  
 $\cos \theta = \cos \theta_0 \left[ 1 + \frac{1}{2} \frac{v}{c} \sin(2\theta_0) - \frac{1}{2} \frac{v^2}{c^2} \sin^2 \theta_0 \right]$   
 where  $\theta_0$  is the angle of reflection,  $\theta$  the angle of incidence and  $\theta'$  is the  
 angle between the direction of drift and the mirror. After substituting the  
 proper values in the general equation, the former transformation was  
 applied.

After determining the effect of the motion upon the angles involved  
 in the light, the exact effect of the motion upon the length of wave path is



found.

Because of the motion, the wave length,  $\lambda'$  of the reflected light differs from  $\lambda$ , the wave length of the incident light according to the equation:

$$\frac{\lambda'}{\lambda} = \frac{\sin i'}{\sin i}$$

After substitution in this equation, the Lorentz transformations are applied. Then, by dividing each component of the two paths by the wave length of the light within that part of the path, we obtained the equations of the number of waves within that path. Having obtained the latter equations, the number of wave lengths in one complete path was subtracted from the number in the other complete path to give the total difference of number of wave lengths in the two paths. This latter equation was given in terms of:  $\lambda_0$ , the fundamental wave length;  $\phi_0$ , the angle between the main arm of the interferometer and the direction of the motion;  $\theta_0$ , the angle between the half-silvered plate and either arm;  $\mu_0$ , an angle proportional to the angle between the normal to one of the mirrors and the arm to which it is fixed, the latter being introduced so that the fringe viewed in the telescope shall be of a convenient width; and the lengths of the two arms  $S_0''$  and  $S_0'$ . The equation of the total difference in the number of fringes is entirely in terms of variables as measured by the moving observer.

Because of the great complexity of this problem it was found necessary to make approximations. Accordingly, all equations were developed only up to the second power of  $v/c$ . Powers of  $v/c$  greater than the second were neglected. This is a justifiable approximation, since the velocity of the earth in its orbit is thirty kilometers per second and the velocity of light is ten

found.

Because of the motion, the wave length  $\lambda'$  of the reflected light

differs from  $\lambda$ , the wave length of the incident light according to the

equation:

$$\frac{\lambda'}{\lambda} = \frac{\sin i}{\sin r}$$

After substitution in this equation, the Lorentz transformation was applied.

Then, by dividing each component of the two paths by the wave length of the

light within that part of the path, we obtained the equations of the number

of waves within that part. We then subtracted the latter equation, the number

of wave lengths in one direction from the number in the

other direction path to give the total difference of number of wave lengths in

the two paths. This latter equation was given in terms of:  $\lambda$ , the wavelength

of wave length  $\lambda'$ , the angle between the main axis of the interferometer and

the direction of the velocity  $v$ , the angle between the half-silvered plate and

either arm  $\lambda'$ , an angle proportional to the angle between the normal to one

of the mirrors and the arm to which it is fixed, the latter being introduced

as that the thing viewed in the telescope shall be of a convenient width;

and the lengths of the two arms  $l_1$  and  $l_2$ . The equation of the total

difference in the number of fringes is entirely in terms of variables as

measured by the moving observer.

Because of the great complexity of this problem it was found neces-

sary to make approximations. As already, all equations were developed only up

to the second power of  $v/c$ . Because of the greater than the second were neg-

lected. This is a justifiable approximation, since the velocity of the earth

is too small to give a noticeable effect on the velocity of light in the



thousand times as great. Hence, since  $v/c$  is of the order of  $10^{-4}$ ,  $(v/c)^3$  is negligible.

Finally, to analyze the solution, the values of  $\theta_0$ ,  $S_0''$ ,  $S_0'$ ,  $\lambda_0$  and  $\mu_0$  used by D. C. Miller<sup>1</sup> were substituted. After assigning an arbitrary value to  $v/c$ , the first and second order terms were plotted and found to be of the form of the experimental curves obtained by D. C. Miller<sup>2</sup>. Thus, making the hypotheses mentioned in the beginning of this abstract, we find that, contrary to the usual opinion, it is possible to derive equations involving both the first and second order effects obtained by the very extensive work of D.C. Miller.

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1. Miller, D.C., Rev. Mod. Phys., 5, 203, (1933)

2. Ibid page 227





# APPENDIX A

$$\sin \epsilon = \cos(\vartheta' + \sigma + i_1' + j_1')$$

Expanding, we obtain:

$$\sin \epsilon = \cos(\vartheta' + \sigma + i_1') \cos j_1' - \sin(\vartheta' + \sigma + i_1') \sin j_1'$$

Substitute equations 43, 45, 55 and 56.

$$\begin{aligned} \sin \epsilon = & \left[ \cos \vartheta_0 - \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \right. \\ & + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) \cos \vartheta_0 + \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 - \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 \right. \\ & - \frac{1}{2} \sin^2 \varphi_0 \cos \vartheta_0 - 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \\ & \left. - 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \right\} \left[ \sin \vartheta_0 + \frac{v}{c} \cos \vartheta_0 \{ \sin(2\vartheta_0 + \varphi_0) + 2 \sin \mu_0 \} \right. \\ & + \frac{v^2}{c^2} \left\{ -\frac{1}{2} \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \sin \vartheta_0 \cos^2 \varphi_0 - \frac{1}{2} \sin \vartheta_0 \sin^2 \varphi_0 \right. \\ & + 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \varphi_0 - 4 \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \\ & + 2 \sin \vartheta_0 \sin \varphi_0 \sin \mu_0 + 2 \cos \vartheta_0 \cos \varphi_0 \sin \mu_0 \\ & \left. \left. - 8 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \mu_0 - 2 \sin \vartheta_0 \sin^2 \mu_0 \right\} \right] \\ & - \left[ \sin \vartheta_0 + \frac{v}{c} \sin(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \right. \\ & + \frac{v^2}{c^2} \left\{ -\frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 + \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 - \frac{1}{2} \sin^2 \varphi_0 \sin \vartheta_0 \right. \\ & \left. - 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 + 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos \vartheta_0 \right\} \\ & \left[ \cos \vartheta_0 - \frac{v}{c} \sin \vartheta_0 \{ \sin(2\vartheta_0 + \varphi_0) + 2 \sin \mu_0 \} \right. \\ & + \frac{v^2}{c^2} \left\{ \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 + \sin \vartheta_0 \sin \varphi_0 \cos \varphi_0 - \frac{1}{2} \cos \vartheta_0 - 2 \sin(\vartheta_0 + \varphi_0) \sin \varphi_0 \right. \\ & + 4 \sin(\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 \sin \varphi_0 + 4 \sin^2(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 \\ & - 2 \sin^2(\vartheta_0 + \varphi_0) \cos \vartheta_0 + 2 \cos \vartheta_0 \sin \varphi_0 \sin \mu_0 - 2 \sin \vartheta_0 \cos \varphi_0 \sin \mu_0 \\ & \left. \left. + 4 \sin(\vartheta_0 + \varphi_0) \sin \mu_0 - 8 \sin(\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 \sin \mu_0 - 2 \cos \vartheta_0 \sin^2 \mu_0 \right\} \right] \end{aligned}$$

# QUESTION 1

$$2\sin \epsilon \leq \cos(\theta + \frac{\pi}{2} + \frac{\pi}{2})$$

Therefore, we obtain:

$$2\sin \epsilon = \cos(\theta + \frac{\pi}{2} + \frac{\pi}{2}) = \sin(\theta + \frac{\pi}{2})$$

Since  $\sin$  is an odd function, it follows that:

$$2\sin \epsilon = [\cos(\theta + \frac{\pi}{2} - \frac{\pi}{2}) - \cos(\theta + \frac{\pi}{2} + \frac{\pi}{2})]$$

$$2\sin \epsilon = \cos(\theta + \frac{\pi}{2} - \frac{\pi}{2}) - \cos(\theta + \frac{\pi}{2} + \frac{\pi}{2}) = \cos(\theta) - \cos(\theta + \pi)$$

$$2\sin \epsilon = \cos(\theta) - (-\cos(\theta)) = \cos(\theta) + \cos(\theta)$$

$$2\sin \epsilon = 2\cos(\theta) \Rightarrow \sin \epsilon = \cos(\theta)$$

$$\sin \epsilon = \cos(\theta) \Rightarrow \epsilon = \frac{\pi}{2} - \theta$$

$$\epsilon = \frac{\pi}{2} - \theta \Rightarrow \theta = \frac{\pi}{2} - \epsilon$$

$$\theta = \frac{\pi}{2} - \epsilon \Rightarrow \theta + \frac{\pi}{2} = \pi - \epsilon$$

$$\theta + \frac{\pi}{2} = \pi - \epsilon \Rightarrow \theta + \frac{\pi}{2} + \frac{\pi}{2} = \pi - \epsilon + \frac{\pi}{2}$$

$$\theta + \pi = \pi - \epsilon + \frac{\pi}{2} \Rightarrow \theta = -\epsilon + \frac{\pi}{2}$$

$$\theta = -\epsilon + \frac{\pi}{2} \Rightarrow \theta + \frac{\pi}{2} = \frac{\pi}{2} - \epsilon + \frac{\pi}{2} = \pi - \epsilon$$

$$\theta + \frac{\pi}{2} = \pi - \epsilon \Rightarrow \theta + \frac{\pi}{2} + \frac{\pi}{2} = \pi - \epsilon + \frac{\pi}{2}$$

$$\theta + \pi = \pi - \epsilon + \frac{\pi}{2} \Rightarrow \theta = -\epsilon + \frac{\pi}{2}$$

$$\theta = -\epsilon + \frac{\pi}{2} \Rightarrow \theta + \frac{\pi}{2} = \frac{\pi}{2} - \epsilon + \frac{\pi}{2} = \pi - \epsilon$$

$$\theta + \frac{\pi}{2} = \pi - \epsilon \Rightarrow \theta + \frac{\pi}{2} + \frac{\pi}{2} = \pi - \epsilon + \frac{\pi}{2}$$

$$\theta + \pi = \pi - \epsilon + \frac{\pi}{2} \Rightarrow \theta = -\epsilon + \frac{\pi}{2}$$

$$\theta = -\epsilon + \frac{\pi}{2} \Rightarrow \theta + \frac{\pi}{2} = \frac{\pi}{2} - \epsilon + \frac{\pi}{2} = \pi - \epsilon$$



$$\sin \epsilon = \sin \vartheta_0 \cos \vartheta_0 - \sin \vartheta_0 \cos \vartheta_0$$

$$\begin{aligned}
 & + \frac{v}{c} [\cos^2 \vartheta_0 \sin(2\vartheta_0 + \varphi_0) + 2 \cos^2 \vartheta_0 \sin \mu_0 - \sin^2 \vartheta_0 \sin(2\vartheta_0 + \varphi_0) \\
 & \quad - \sin(2\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 + \sin(2\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 + 2 \sin^2 \vartheta_0 \sin \mu_0] \\
 & + \frac{v^2}{c^2} [-\frac{1}{2} \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 + \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 \cos^2 \varphi_0 \\
 & \quad - \frac{1}{2} \sin \vartheta_0 \sin^2 \varphi_0 \cos \vartheta_0 + 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \sin \varphi_0 \\
 & \quad - 4 \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 + 2 \sin \vartheta_0 \cos \vartheta_0 \sin \varphi_0 \sin \mu_0 \\
 & \quad + 2 \cos^2 \vartheta_0 \cos \varphi_0 \sin \mu_0 - 8 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \sin \mu_0 \\
 & \quad - 2 \sin \vartheta_0 \cos \vartheta_0 \sin^2 \mu_0 - \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \\
 & \quad - 2 \sin(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \mu_0 + \frac{1}{2} \cos^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \\
 & \quad + \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 - \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \\
 & \quad - \frac{1}{2} \sin^2 \varphi_0 \sin \vartheta_0 \cos \vartheta_0 - 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos^2 \vartheta_0 \\
 & \quad - 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 + \frac{1}{2} \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \\
 & \quad - \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 + \frac{1}{2} \sin^2 \varphi_0 \sin \vartheta_0 \cos \vartheta_0 \\
 & \quad + 2 \sin(\vartheta_0 + \varphi_0) \cos(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos^2 \vartheta_0 - 4 \sin(2\vartheta_0 + \varphi_0) \cos(2\vartheta_0 + \varphi_0) \cos^2 \vartheta_0 \\
 & \quad + \sin^2(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 + 2 \sin(2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \sin \mu_0 \\
 & \quad - \frac{1}{2} \cos^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 - \sin^2 \vartheta_0 \sin \varphi_0 \cos \varphi_0 \\
 & \quad + \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 + 2 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \sin \varphi_0 \\
 & \quad - 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \sin \varphi_0 - 4 \sin^2(\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \cos \vartheta_0 \\
 & \quad + 2 \sin^2(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 - 2 \cos \vartheta_0 \sin \varphi_0 \sin \vartheta_0 \sin \mu_0 \\
 & \quad + 2 \sin^2 \vartheta_0 \cos \varphi_0 \sin \mu_0 - 4 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \sin \mu_0 \\
 & \quad + 8 \sin(\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos^2 \vartheta_0 \sin \mu_0 + 2 \sin \vartheta_0 \cos \vartheta_0 \sin^2 \mu_0] + \dots
 \end{aligned}$$

After making the indicated cancellations, the above equation

becomes:





$$\begin{aligned}
\sin \epsilon &= 2 \frac{v}{c} \sin \mu_0 \\
&+ \frac{v^2}{c^2} \left[ \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 \cos^2 \varphi_0 - 2 \sin^2 (\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \right. \\
&\quad + 2 \cos \varphi_0 \sin \mu_0 + \sin (\vartheta_0 + \varphi_0) \cos (\vartheta_0 + \varphi_0) \sin^2 \vartheta_0 \\
&\quad - \frac{1}{2} \sin^2 \varphi_0 \sin \vartheta_0 \cos \vartheta_0 - 4 \sin (2\vartheta_0 + \varphi_0) \cos (2\vartheta_0 + \varphi_0) \\
&\quad + \frac{1}{2} \sin^2 (2\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 - \frac{1}{2} \cos^2 (\vartheta_0 + \varphi_0) \sin \vartheta_0 \cos \vartheta_0 \\
&\quad - \sin^2 \vartheta_0 \sin \varphi_0 \cos \varphi_0 + 2 \sin (\vartheta_0 + \varphi_0) \sin \vartheta_0 \sin \varphi_0 \\
&\quad \left. - 4 \sin (\vartheta_0 + \varphi_0) \sin \vartheta_0 \sin \mu_0 \right] + \dots
\end{aligned}$$

Upon further simplification, this reduces to:

$$\begin{aligned}
\sin \epsilon &= 2 \frac{v}{c} \sin \mu_0 + \frac{v^2}{c^2} \left[ \frac{1}{2} \sin \vartheta_0 \cos \vartheta_0 \{ \sin^2 (2\vartheta_0 + \varphi_0) - \sin^2 (\vartheta_0 + \varphi_0) \} \right. \\
&\quad - 4 \sin (2\vartheta_0 + \varphi_0) \cos (2\vartheta_0 + \varphi_0) \\
&\quad \left. + 2 \cos (2\vartheta_0 + \varphi_0) \sin \mu_0 \right] + \dots
\end{aligned}$$

Eqn. (58)





## APPENDIX B

In deriving the equations for the length of the paths  $D_1'$  and  $D_2'$ , it was tacitly assumed that the ray reflected from the plate M was incident upon the mirror  $M'$  at its center and that the ray reflected from the mirror  $M'$  was incident upon the plate M at its center. We shall now prove these facts.

In figure IX angles and distances corresponding to angles and distances in figure I have been designated by the same letters as in that figure. Let us assume that the ray  $ct_1'$  is incident upon the mirror  $M'$  at any arbitrary point  $C'$ . The angle  $\psi$  will still be defined as the angle between  $S'$  and the ray  $ct_1'$ . Also,  $i_2$  will retain its definition as the angle of incidence of the ray at the mirror  $M'$ . Construct  $AC''$  perpendicular to the mirror  $M'$ , then, in triangle  $AC'C''$ :

$$\cos i_2 = \frac{AC''}{AC'} = \frac{AC''}{ct_1'}$$

In triangle  $AE''C''$ :

$$\text{Angle } AC''E'' = \sigma$$

Therefore:

$$AC'' = (S' - EE'') \cos \sigma.$$

To find  $EE''$  construct  $EE'$  perpendicular to  $CC''$ . Then, angle  $E'EE''$  equals  $\sigma$ .

Also, in the right triangle  $CEE'$ :

$$\text{Angle } CEE' = [\pi - (\vartheta + \vartheta' + \varphi + \sigma)]$$

Therefore:

$$\begin{aligned} EE' &= \nu t_1' \cos [\pi - (\vartheta + \vartheta' + \varphi + \sigma)] \\ &= -\nu t_1' \cos (\vartheta + \vartheta' + \varphi + \sigma) \end{aligned}$$

# APPENDIX B

In deriving the equations for the length of the paths  $D_1$  and  $D_2$ , it was tacitly assumed that the ray reflected from the plate  $M$  was incident upon the mirror  $N$  at the center and that the ray reflected from the mirror  $N$  was incident upon the plate  $M$  at its center. We shall now prove these

facts.

In Figure 1, angles and distances corresponding to angles and distances in Figure 1 have been designated by the same letters as in that figure. Let us assume that the ray  $cd$  is incident upon the mirror  $N$  at any arbitrary point  $C$ . The angle  $\psi$  will still be defined as the angle between  $d'$  and the ray  $cd$ . Also,  $d'$  will retain its definition as the angle of incidence of the ray at the mirror  $N$ . Construct  $AC$  perpendicular

to the mirror  $N$ , then, in triangle  $AC'C$ :

$$\cos \psi = \frac{AC''}{AC'} = \frac{AC''}{d'}$$

In triangle  $AE'C$ :

$$\angle A'CE = \angle AC'E = \alpha$$

Therefore:

$$AC'' = (d' - EE') \cos \alpha$$

To find  $EE'$  construct  $EE'$  perpendicular to  $CC'$ . Then, angle  $EE'C$  equals  $\alpha$ .

Also, in the right triangle  $CEE'$ :

$$\angle CEE' = [180 - (90 + \alpha + \alpha)]$$

Therefore:

$$EE' = d' \cos [180 - (90 + \alpha + \alpha)]$$

$$= -d' \cos (90 + \alpha + \alpha)$$



Now, in triangle  $EE'E''$  :

$$EE'' = \frac{EE'}{\cos \sigma}$$

$$= - \frac{vt' \cos(\theta + \theta' + \phi + \sigma)}{\cos \sigma}$$

Substituting this expression for  $EE''$  in the equation for  $AC''$ , we obtain:

$$AC'' = S' \cos \sigma + vt' \cos(\theta + \theta' + \phi + \sigma)$$

In the first equation, replace  $AC''$  by this last expression, this gives:

$$ct' = \frac{S' \cos \sigma + vt' \cos(\theta + \theta' + \phi + \sigma)}{\cos i_2}$$

Solving for  $t'$ , we get:

$$t' = \frac{S' \cos \sigma}{c \cos i_2 - v \cos(\theta + \theta' + \phi + \sigma)}$$

Multiply both sides by  $c$  and replace  $ct'$  by  $D'_1$ . This gives:

$$D'_1 = \frac{S' \cos \sigma}{\cos i_2 - \frac{v}{c} \cos(\theta + \theta' + \phi + \sigma)}$$

From figure IX, we see that:

$$\psi = i_2 - \sigma, \quad \text{Also} \quad \psi = i'_1 + \theta' - \frac{\pi}{2}$$

Therefore,

$$i_2 = \theta' + \sigma + i'_1 - \frac{\pi}{2}$$

and

$$\cos i_2 = \sin(\theta' + \sigma + i'_1)$$

Expand and substitute the latter expression in the equation for  $D'_1$ . Also, divide numerator and denominator by  $\cos \sigma$  and expand the  $v/c$  term. This yields:

$$D'_1 = \frac{S'}{\sin(\theta' + i'_1) + \cos(\theta' + i'_1) \frac{\sin \sigma}{\cos \sigma} - \frac{v}{c} \cos(\theta + \theta' + \phi) + \frac{v}{c} \sin(\theta + \theta' + \phi) \frac{\sin \sigma}{\cos \sigma}}$$

Now, in order to find  $\psi$ :

$$EE'' = \frac{EE'}{\cos \alpha}$$

$$= - \frac{v'(\cos(\beta + \gamma + \alpha))}{\cos \alpha}$$

Substituting this expression for  $EE''$  in the equation for  $\psi$ , we obtain:

$$AC'' = 2' \cos \alpha + v'(\cos(\beta + \gamma + \alpha))$$

In the first equation, replace  $AC''$  by this last expression, this gives:

$$C' = \frac{2' \cos \alpha + v'(\cos(\beta + \gamma + \alpha))}{\cos \alpha}$$

Solving for  $\alpha$ , we get:

$$\alpha' = \frac{2' \cos \alpha}{\cos \alpha - v' \cos(\beta + \gamma + \alpha)}$$

Multiply both sides by  $\cos \alpha$  and replace  $\alpha'$  by  $\psi$ . This gives:

$$D' = \frac{2' \cos \alpha}{\cos \alpha - \frac{v'}{C} \cos(\beta + \gamma + \alpha)}$$

From figure 11, we see that:

$$\psi = \alpha' - \alpha, \quad \text{also} \quad \psi = \beta + \gamma - \frac{\pi}{2}$$

Therefore,

$$\alpha' = \beta + \gamma - \frac{\pi}{2} - \alpha$$

and

$$\cos \alpha' = \sin(\beta + \gamma + \alpha)$$

Expand and substitute the latter expression in the equation for  $D'$ . Also,

divide numerator and denominator by  $\cos \alpha$  and expand the  $v'/C$  term. This

gives:

$$D' = \frac{2'}{\cos \alpha} \frac{\sin(\beta + \gamma + \alpha) + \cos(\beta + \gamma + \alpha) \frac{v'}{C} \sin \alpha}{\cos \alpha - \frac{v'}{C} \cos(\beta + \gamma + \alpha)}$$



In substituting the equations 15, 16 and 32, the second and the last terms of the denominator have as their coefficients powers of  $v/c$  greater than the second therefore, they will be neglected. What remains in the equation is the same as the expression for  $D'_1$  found on page 56 of the main body of this thesis.

Thus, since the two expressions for  $D'_1$  are the same, it must follow that the points  $C'$  and  $C$  are coincident and the ray actually strikes at the center of the mirror  $M'$ .

Having proved that the ray reflected from the plate  $M$  is incident upon the center of the mirror  $M'$ , let us represent the ray reflected from  $M'$  by the line  $CH'$ , where  $H'$  represents any point on the plate  $M$ . Let  $K$  represent the position of the plate  $M$  at the time  $t'_1$  and  $J$  the point of intersection of the ray  $ct'_1$  and the line  $AH$ .

From the figure IX, we see that:

$$ct'_1 = D'_2 = CJ + JH'$$

Now, in the triangle  $CJK$  :

$$\frac{\sin(\vartheta + \vartheta' + \varphi)}{CJ} = \frac{\sin[\pi - (\vartheta + \vartheta' + \varphi + \sigma + i'_2)]}{s'} = \frac{\sin(i'_2 + \sigma)}{KJ}$$

Therefore:

$$CJ = \frac{s' \sin(\vartheta + \vartheta' + \varphi)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i'_2)}$$

and:

$$KJ = \frac{s' \sin(i'_2 + \sigma)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i'_2)}$$

From the figure IX, we observe that:

$$JH = vt'_2 - KJ = vt'_2 - \frac{s' \sin(i'_2 + \sigma)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i'_2)}$$

In substituting the equations 15, 16 and 17, the second and the last terms of the denominator have as their coefficients terms of  $v/c$  greater than the second term, they will be neglected. That remains in the equation is the same as the expression for  $\beta'$  found on page 36 of the main body of this thesis.

Thus, since the two expressions for  $\beta'$  are the same, it must follow that the points  $B'$  and  $B$  are coincident and the ray actually strikes at the center of the mirror  $B'$ .

Having proved that the ray reflected from the plate  $M$  is incident upon the center of the mirror  $B'$ , let us represent the ray reflected from  $B'$  by the line  $CB'$ , where  $B'$  represents any point on the plate  $M$ . Let  $E$  represent the position of the plate  $M$  at the time  $t_1$  and  $t_2$  the point of intersection of the ray  $CB'$  and the line  $AE$ .

From the figure 15, we see that:

$$CB' = D' = CB + BH'$$

Now, in the triangle  $CB'E$ :

$$\frac{CB}{\sin(\theta + \phi + \psi)} = \frac{CE}{\sin(\theta + \phi + \psi + \frac{1}{2}\pi)} = \frac{BE}{\sin(\frac{1}{2}\pi)}$$

Therefore:

$$CB = \frac{CE \sin(\frac{1}{2}\pi + \psi)}{\sin(\theta + \phi + \psi + \frac{1}{2}\pi)}$$

and:

$$KB = \frac{CE \sin(\frac{1}{2}\pi + \psi)}{\sin(\theta + \phi + \psi + \frac{1}{2}\pi)}$$

From the figure 15, we observe that:

$$BH = \sqrt{t_1^2 - KB^2} = \sqrt{t_1^2 - \frac{CE^2 \sin^2(\frac{1}{2}\pi + \psi)}{\sin^2(\theta + \phi + \psi + \frac{1}{2}\pi)}}$$



In the triangle JHH' we have, by the sine law:

$$JH' = \frac{JH \sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')}$$

Substituting the previous equation for JH', we obtain:

$$JH' = \frac{vt_2' \sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')} - \frac{s' \sin(i_2' + \sigma) \sin(\vartheta + \varphi)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2') \sin(\vartheta' + \sigma + i_2')}$$

Now, in the equation for  $ct_2'$  substitute the equations for CJ and JH'. This gives:

$$ct_2' = \frac{s' \sin(\vartheta + \vartheta' + \varphi)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')} + \frac{vt_2' \sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')} - \frac{s' \sin(i_2' + \sigma) \sin(\vartheta + \varphi)}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2') \sin(\vartheta' + \sigma + i_2')}$$

Solving for  $t_2'$ , we get:

$$t_2' = \frac{s'}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')} \left[ \frac{\sin(\vartheta + \vartheta' + \varphi) \sin(\vartheta' + \sigma + i_2') - \sin(i_2' + \sigma) \sin(\vartheta + \varphi)}{c \sin(\vartheta' + \sigma + i_2') - v \sin(\vartheta + \varphi)} \right]$$

Multiply both sides of the equation by  $c$  and divide the numerator and the denominator of the right hand member by  $\sin(\vartheta' + \sigma + i_2')$ . This gives:

$$D_2' = \frac{s'}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')} \left[ \frac{\sin(\vartheta + \vartheta' + \varphi) - \frac{\sin(i_2' + \sigma) \sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')}}{1 - \frac{v}{c} \frac{\sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')}} \right]$$

Expanding the denominator, we get:

$$D_2' = \frac{s'}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')} \left[ \sin(\vartheta + \vartheta' + \varphi) - \frac{\sin(i_2' + \sigma) \sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')} \right] \left[ 1 + \frac{v}{c} \frac{\sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')} + \frac{v^2}{c^2} \frac{\sin^2(\vartheta + \varphi)}{\sin^2(\vartheta' + \sigma + i_2')} + \dots \right]$$

Multiplying the two brackets and observing that  $\sin(i_2' + \sigma)$  is of the order of

$v/c$ , we get:  $D_2' = \frac{s'}{\sin(\vartheta + \vartheta' + \varphi + \sigma + i_2')} \left[ \sin(\vartheta + \vartheta' + \varphi) + \frac{\sin(\vartheta + \varphi)}{\sin(\vartheta' + \sigma + i_2')} \left\{ \frac{v}{c} \sin(\vartheta + \vartheta' + \varphi) - \sin(i_2' + \sigma) \right\} \right. \\ \left. + \frac{\sin^2(\vartheta + \varphi)}{\sin^2(\vartheta' + \sigma + i_2')} \left\{ \frac{v^2}{c^2} \sin^2(\vartheta + \vartheta' + \varphi) - \frac{v}{c} \sin(i_2' + \sigma) \right\} + \dots \right]$

In the triangle  $HM'$ , we have, by the sine law:

$$HM' = \frac{H \sin(\theta + \psi)}{\sin(\theta' + \psi)}$$

Substituting the previous equation for  $HM$ , we obtain:

$$HM' = \frac{H \sin(\theta + \psi)}{\sin(\theta' + \psi)} - \frac{H \sin(\theta + \psi) \sin(\theta' + \psi)}{\sin(\theta + \psi) \sin(\theta' + \psi) \sin(\theta' + \psi)}$$

Now, in the equation for  $ct'_1$ , substituting the equations for  $HM$  and  $HM'$ . This gives:

$$ct'_1 = \frac{H \sin(\theta + \psi)}{\sin(\theta' + \psi)} + \frac{H \sin(\theta + \psi) \sin(\theta' + \psi)}{\sin(\theta + \psi) \sin(\theta' + \psi) \sin(\theta' + \psi)}$$

Solving for  $\theta'_1$ , we get:

$$\theta'_1 = \frac{H \sin(\theta + \psi)}{c \sin(\theta' + \psi) \sin(\theta' + \psi)} \left[ \frac{\sin(\theta + \psi) \sin(\theta' + \psi) - \sin(\theta + \psi) \sin(\theta' + \psi)}{\sin(\theta + \psi) \sin(\theta' + \psi)} \right]$$

Multiply both sides of the equation by  $c$  and divide the numerator and the denominator of the right hand member by  $\sin(\theta' + \psi)$ . This gives:

$$D'_1 = \frac{H \sin(\theta + \psi)}{c \sin(\theta' + \psi) \sin(\theta' + \psi)} \left[ \frac{\sin(\theta + \psi) \sin(\theta' + \psi) - \sin(\theta + \psi) \sin(\theta' + \psi)}{\sin(\theta + \psi) \sin(\theta' + \psi)} \right]$$

Expanding the denominator, we get:

$$D'_1 = \frac{H \sin(\theta + \psi)}{c \sin(\theta' + \psi) \sin(\theta' + \psi)} \left[ \frac{\sin(\theta + \psi) \sin(\theta' + \psi) - \sin(\theta + \psi) \sin(\theta' + \psi)}{\sin(\theta + \psi) \sin(\theta' + \psi)} \right]$$

Multiplying the two brackets and observing that  $\sin(\theta' + \psi)$  is of the order of

$$D'_1 = \frac{H \sin(\theta + \psi)}{c \sin(\theta' + \psi) \sin(\theta' + \psi)} \left[ \frac{\sin(\theta + \psi) \sin(\theta' + \psi) - \sin(\theta + \psi) \sin(\theta' + \psi)}{\sin(\theta + \psi) \sin(\theta' + \psi)} \right]$$



Expanding  $\sin(i_2' + \sigma)$  and substituting equations 15, 16, 39 and 40; also, replacing  $\sin(\theta + \theta' + \phi)$  by equation 7, we find that the two braces reduce to zero. Therefore:

$$D_2' = s' \frac{\sin(\theta + \theta' + \phi)}{\sin(\theta + \theta' + \phi + \sigma + i_2')}$$

It will be observed that this latter expression is the same as that for  $D_2'$  at the top of page 58.

Thus, since the two expressions for  $D_2'$  are the same, it must follow that the points  $H'$  and  $H$  are coincident. Hence, the ray actually strikes at the center of the plate  $M$ .





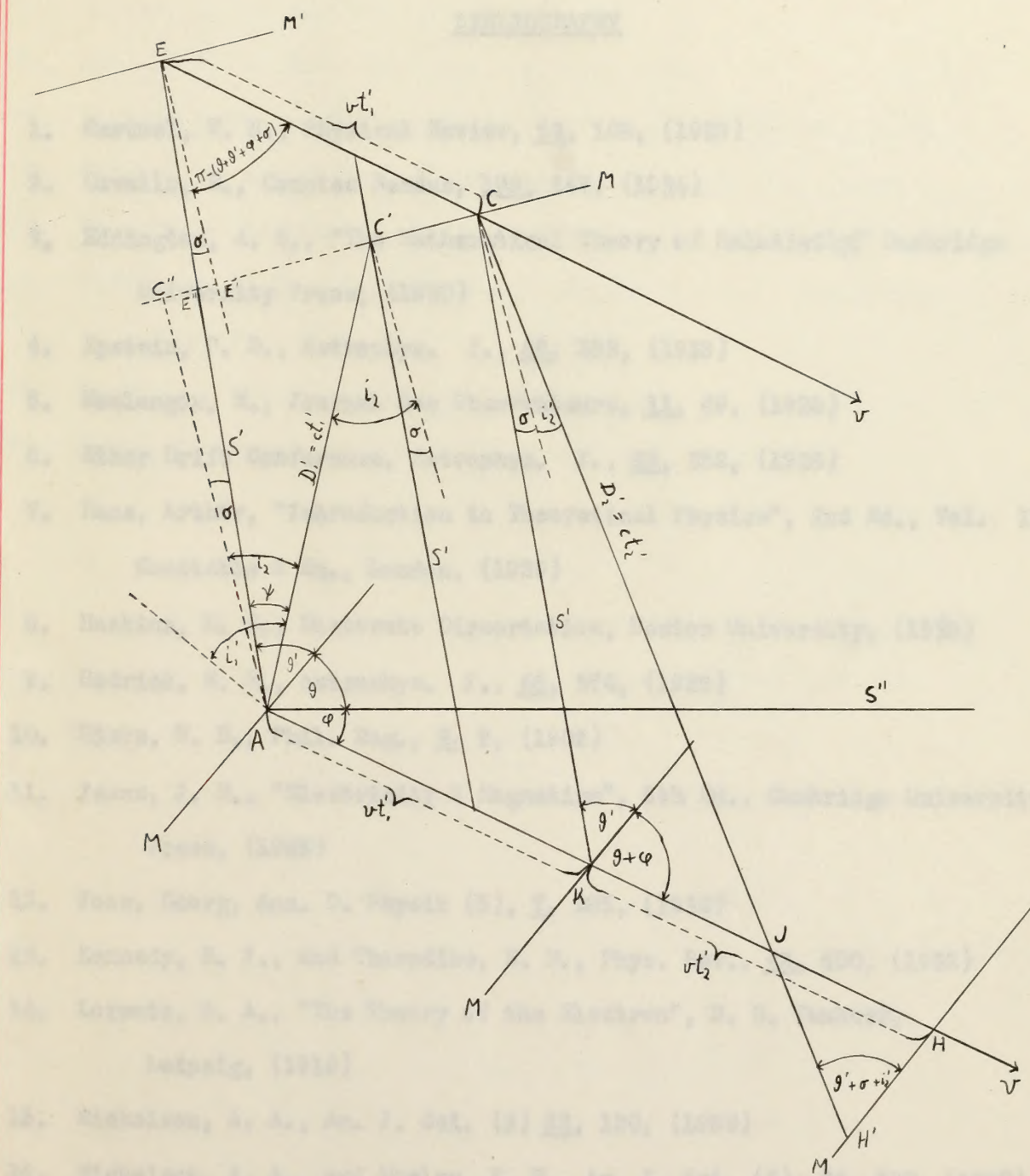


Figure IX

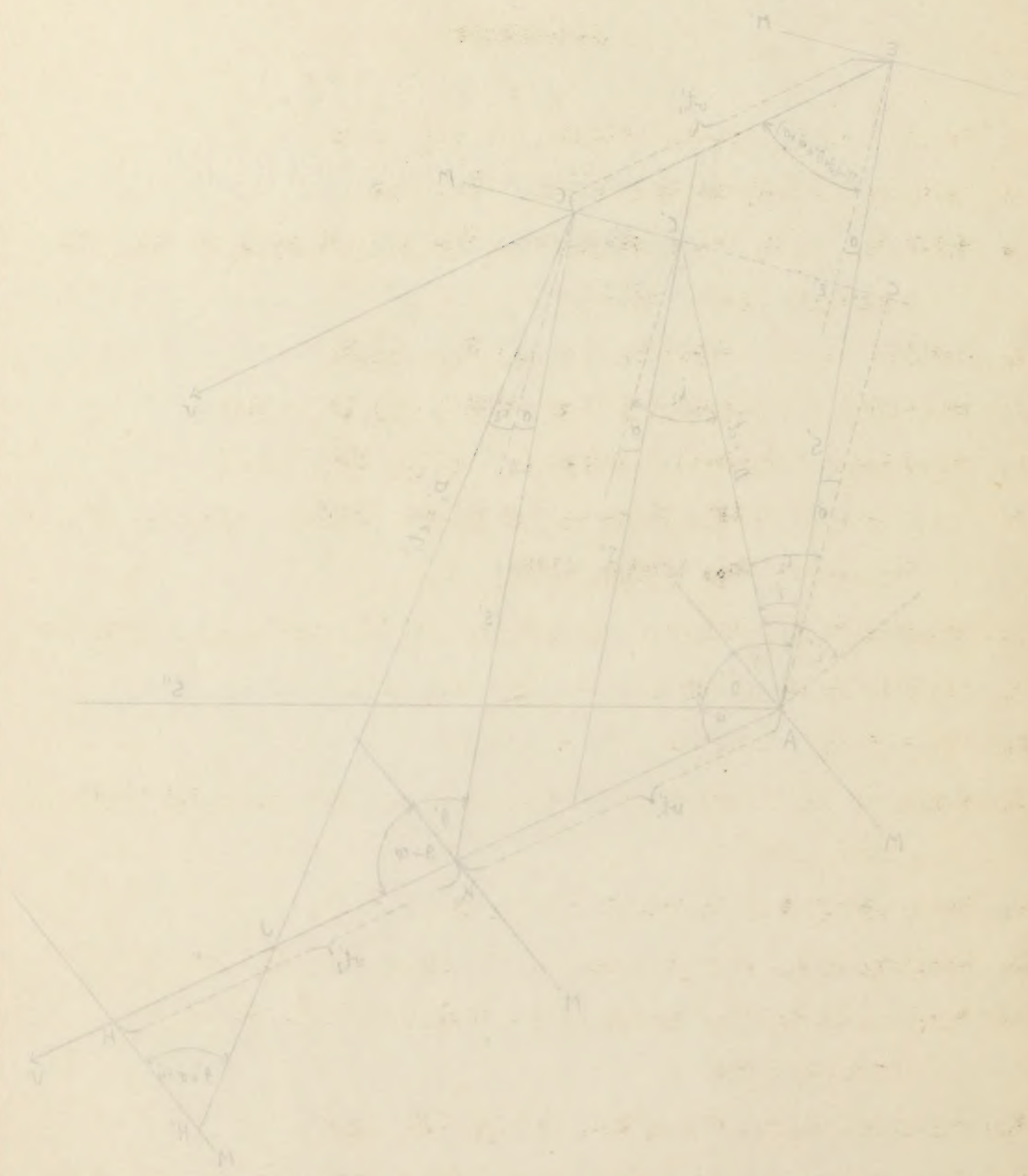


Figure IX



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AUTOBIOGRAPHY

The author was born July 3, 1913, in Boston, Massachusetts, the son of David and Celia Lisman. He was graduated from the Wellington Grammar School in Cambridge in 1926, and from the Dorchester High School in 1930.

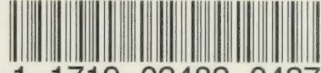
In June 1934 he received the degree of Bachelor of Science in Education from Boston University; his major fields of study having been education, physics, mathematics and chemistry. In June 1935 he received the degree of Master of Arts from Boston University. During that year, he was an assistant in the Boston University physics department. In the following four academic years, he took part time work at Harvard University and Boston University leading to the degree of Doctor of Philosophy.

During all the years that he has been attending the University with the exception of the year 1934-35, he has been teaching in private schools in Lowell and in Boston. He was married to Rachel Lewit on April 3, 1938.





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